Summary of Notes on Finite-Deformation of Isotropic Elastic-Viscoplastic Materials

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1 Summary of three-dimensional large-deformation rate-dependent elastic-viscoplastic theory

1. The **Kinematical** Kröner-Lee decomposition

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p, \qquad \text{with} \qquad \det \mathbf{F}^{\mathrm{p}} = 1, \tag{1}$$

in which $\mathbf{F}$ is the deformation gradient, while $\mathbf{F}^e$ and $\mathbf{F}^p$ are the elastic and plastic distortions.

2. The polar decomposition

$$\mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e \tag{2}$$

of the elastic distortion $\mathbf{F}^e$, and the spectral decomposition

$$\mathbf{U}^e = \sum_{i=1}^{3} \lambda_i^e \mathbf{r}_i^e \otimes \mathbf{r}_i^e, \tag{3}$$

of the right elastic stretch tensor $\mathbf{U}^e$. Here $\{\lambda_i^e\}$ and $\{\mathbf{r}_i^e\}$ are, respectively, the positive eigenvalues (principal elastic stretches) and orthonormal eigenvectors of $\mathbf{U}^e$. From this we define the **Hencky strain (logarithmic strain measure)** based on the right elastic orthonormal basis

$$\mathbf{E}^e = \ln \mathbf{U}^e = \sum_{i=1}^{3} (\ln \lambda_i^e)\, \mathbf{r}_i^e \otimes \mathbf{r}_i^e. \tag{4}$$

3. Next, for isotropic materials, we consider a specialized equation for the **elastic free energy**

$$\psi = G|\mathbf{E}^e|^2 + \frac{1}{2}(K - \frac{2}{3}G)(\mathrm{tr}\mathbf{E}^e)^2, \tag{5}$$

which yields the symmetric **Mandel stress (driving stress for plastic flow)**

$$\mathbf{M}^e = 2G\mathbf{E}^e + (K - \frac{2}{3}G)(\mathrm{tr}\,\mathbf{E}^e)\mathbf{1} \tag{6}$$

and the **Cauchy stress** is obtained using

*Refer to Professor Lallit Anand's 2.073 plasticity notes for detailed theory on which this summary is based.

$$\mathbf{T} = J^{e-1}\mathbf{R}^e\mathbf{M}^e\mathbf{R}^{e\top}, \quad J^e = \det \mathbf{F}^e = \det \mathbf{F} \text{ (since } \det \mathbf{F}^p = 1). \tag{7}$$

Next, we define an **equivalent shear stress** as

$$\bar{\tau} \stackrel{\text{def}}{=} \frac{1}{\sqrt{2}} \left| \mathbf{M}^e_0 \right|, \tag{8}$$

noting that this can be thought of as an invariant of $\mathbf{M}^e$.

4. **Flow Rule:** $\mathbf{F}^p$ evolves as

$$\dot{\mathbf{F}}^p = \mathbf{D}^p\mathbf{F}^p, \tag{9}$$

with

$$\mathbf{D}^p = \nu^p \frac{\mathbf{M}^e_0}{2\bar{\tau}}, \tag{10}$$

where

$$\nu^p \stackrel{\text{def}}{=} \sqrt{2}|\mathbf{D}^p|. \tag{11}$$

Equivalent plastic shear strain rate ν^p is given by a flow function

$$\nu^p = f(\bar{\tau}, S) \quad \text{with} \quad \nu^p = 0 \;\text{ when }\; \bar{\tau} = 0, \tag{12}$$

where S *is an internal variable which may evolve with deformation.*

An example of a commonly used rate-sensitive flow rule function is the **power-law function**

$$\nu^p = \nu_0 \left(\frac{\bar{\tau}}{S} \right)^{(1/m)}, \tag{13}$$

where ν_0 is a parameter with units of 1/time, S has units of stress (flow strength) and is assumed to be a constant (for simplicity we neglect evolution of S), and m is a rate sensitivity parameter. As the value of m decreases, the material becomes more and more rate-independent. The inverted form of (13) is

$$\bar{\tau} = S \left(\frac{\nu^p}{\nu_0} \right)^m. \tag{14}$$

The power-law function allows one to characterize nearly *rate-independent behavior* when m is very small.

Note that in contrast to the rate-independent theory, the plastic strain-rate is nonzero whenever the stress is nonzero: there is no elastic range in which the response of the material is purely elastic, and there are no considerations of a yield condition, a consistency condition, loading/unloading conditions, and so forth.

Note for a special case of $m = 1$, we obtain a flow rule that follows Newtonian viscosity.

We can also introduce a *rate-independent initial-yield* with $\tau_e \stackrel{\text{def}}{=} \bar{\tau} - \tau_y$, and writing the power-law form of the flow rule as:

$$\nu^p = \begin{cases} 0 & \text{if} \quad \tau_e \leq 0, \\ \nu_0 \left(\dfrac{\tau_e}{S} \right)^{(1/m)} & \text{if} \quad \tau_e > 0. \end{cases} \tag{15}$$

For a special case where $m = 1$ (Newtonian viscosity), the above flow rule can be used for a **Bingham material** as we have

$$\nu^p = \begin{cases} 0 & \text{if} \quad \bar{\tau} - \tau_y \leq 0, \\ \nu_0 \left(\dfrac{\bar{\tau} - \tau_y}{S} \right) & \text{if} \quad \bar{\tau} - \tau_y > 0, \end{cases} \tag{16}$$

where $\frac{S}{\nu_0}$ is the Newtonian viscosity.

2 Summary of one-dimensional large-deformation rate-dependent elastic-viscoplastic theory

This section presents a summary of an approximate one-dimensional version of the theory. *The approximation is primarily in that we cannot account for Poisson's-type lateral contractions, and attendant volume changes, in a one-dimensional setting.* The underlying constitutive equations relate the following basic fields:

$U > 0$,	stretch (defined as l/l_0, ratio of deformed length l with the original length l_0),
$U = U^e U^p$	elastic-plastic decomposition of U,
U^e	elastic part of the stretch,
U^p,	plastic part of the stretch,
σ,	Cauchy stress.

1. **Strain**

$$\epsilon^e = \ln U^e \tag{17}$$

2. **Free energy**

$$\psi^e = \frac{1}{2} E \, (\epsilon^e)^2 \tag{18}$$

3. **Cauchy stress**

$$\sigma = E \, \epsilon^e \tag{19}$$

4. **Flow rule**

$$\dot{U}^p = D^p U^p, \qquad D^p = \dot{\epsilon}^p \mathrm{sign}(\sigma), \qquad \text{with} \qquad \dot{\epsilon}^p \geq 0, \tag{20}$$

where **equivalent tensile plastic strain rate** $\dot{\epsilon}^p$ needs an evolution equation. With $\bar{\sigma} \stackrel{\text{def}}{=} |\sigma|$, the power-law function for the flow rule is

$$\dot{\epsilon}^p = \dot{\epsilon}_0 \left(\frac{\bar{\sigma}}{S} \right)^{(1/m)}, \tag{21}$$

where ϵ_0 is a parameter with units of 1/time, S has units of stress (it can evolve with deformation but for simplicity we neglect evolution of S and take it to be a constant), and m is a rate sensitivity parameter. $m \to 0$, is the rate-independent limit.

3 Relationship between material parameters in 1D model with material parameters in 3D model

In the above equations, except for the parameters ν_0 and S, the values of the one-dimensional material parameters are unchanged when used in the three-dimensional equations. Since the stress-power should be same for the three-dimensional and one-dimensional cases, we have

$$\bar{\tau}\nu = \bar{\sigma}|\dot{\epsilon}|. \tag{22}$$

Also note that the equivalent shear stress $\bar{\tau}$ is the mises-stress and for the one-dimensional case it relates to tensile stress $\bar{\sigma}$ as

$$\bar{\tau} = \frac{1}{\sqrt{3}} \bar{\sigma}, \tag{23}$$

which with (22) gives

$$\nu = \sqrt{3} \, |\dot{\epsilon}|. \tag{24}$$

The parameters ν_0 and S may be converted from the one-dimensional compression/tension form to the three-dimensional shear form using

$$\nu_0 = \sqrt{3}\,\dot{\epsilon}_0, \quad S^{(3D)} = \frac{1}{\sqrt{3}}\,S^{(1D)}. \tag{25}$$

4 One-dimensional small-deformation rate-dependent elastic-viscoplastic theory

For small deformations, we don't need to distinguish between the deformed configuration and the original configuration (an approximation valid only for small deformations, typically up to the strains of the order of $\approx 0.2\%$). We assume that the total strain ϵ may be additively decomposed as

$$\epsilon = \epsilon^e + \epsilon^p, \tag{26}$$

and call ϵ^e and ϵ^p the elastic and plastic parts of the strain, respectively. Hence, the strain rate also admits the decomposition

$$\dot{\epsilon} = \dot{\epsilon^e} + \dot{\epsilon^p}. \tag{27}$$

The elastic stress-strain relation giving Cauchy stress is now

$$\sigma = E\,\epsilon^e = E\,(\epsilon - \epsilon^p). \tag{28}$$

For rest of the analysis we can use the flow rule relation in section 2.