2.073 Notes on finite-deformation theory of isotropic elastic-viscoplastic solids

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1 Introduction

The purpose of these notes is to develop a constitutive theory for finite-deformation isotropic plasticity with isotropic strain-hardening under *isothermal conditions* at fixed temperature $\vartheta =$ constant, and in the absence of temperature gradients.

2 Kinematics

The Kröner–Lee decomposition

Physical considerations of the mechanisms of elastic–plastic deformation of a solid associate a notion of a *material structure* with the solid that may be stretched and rotated, together with a notion of *defects* capable of flowing through that structure. In the finite deformation theory of elastic-plastic solids, we mathematize this picture with a kinematical constitutive assumption that the deformation gradient $\mathbf{F}(\mathbf{X})$ admit a multiplicative decomposition

$$\mathbf{F}(\mathbf{X}) = \mathbf{F}^e(\mathbf{X})\mathbf{F}^p(\mathbf{X}), \tag{2.1}$$

in which:

(i) $\mathbf{F}^e(\mathbf{X})$ represents the local deformation of material (near $\mathbf{X}$) due to stretch and rotation of the material structure;

(ii) $\mathbf{F}^p(\mathbf{X})$ represents the local deformation of material (near $\mathbf{X}$) due to the flow of defects through that structure.

We refer to (2.1) as the **Kröner–Lee decomposition**.[1] Consistent with our assumption that

$$J = \det \mathbf{F} > 0,$$

we assume that

$$J^e = \det \mathbf{F}^e > 0, \qquad J^p = \det \mathbf{F}^p > 0, \tag{2.2}$$

so that both $\mathbf{F}^e$ and $\mathbf{F}^p$ are invertible.

In discussing the Kröner–Lee decomposition it is important to fully understand the differences between the tensor fields $\mathbf{F}$, $\mathbf{F}^e$, and $\mathbf{F}^p$. First of all, while $\mathbf{F} = \nabla\boldsymbol{\chi}$ is the gradient of a point field, in general there is no point field $\boldsymbol{\chi}^p$ such that $\mathbf{F}^p = \nabla\boldsymbol{\chi}^p$, nor is there a point field $\boldsymbol{\chi}^e$ such that $\mathbf{F}^e = \nabla\boldsymbol{\chi}^e$.

[1] Kröner (1962), Lee (1969).

Thus we can at most describe the physical nature of the tensor fields $\mathbf{F}^e$ and $\mathbf{F}^p$ through their pointwise mapping properties as *linear transformations.* With this in mind, consider the formal relation

$$\mathbf{dx} = \mathbf{F}\mathbf{dX}. \tag{2.3}$$

Equation (2.3) represents a mapping of an infinitesimal neighborhood of $\mathbf{X}$ in the undeformed body to an infinitesimal neighborhood of $\mathbf{x} = \chi_t(\mathbf{X})$ in the deformed body and characterizes $\mathbf{F}$ pointwise as a linear transformation of material vectors to spatial vectors. As is clear from (2.3) and the Kröner–Lee decomposition (2.1),

$$\mathbf{dx} = \mathbf{F}^e\mathbf{F}^p\mathbf{dX}. \tag{2.4}$$

For want of a better notation, let $\mathbf{dl}$ denote $\mathbf{F}^p\mathbf{dX}$,[2]

$$\mathbf{dl} = \mathbf{F}^p\mathbf{dX},$$

so that, by (2.4),

$$\mathbf{dx} = \mathbf{F}^e\mathbf{dl}.$$

Thus, the output of the linear transformation $\mathbf{F}^p$ must coincide with the input of the linear transformation $\mathbf{F}^e$; that is,

$$\text{the range of } \mathbf{F}^p = \text{the domain of } \mathbf{F}^e. \tag{2.5}$$

We refer to this common space as the **structural space** or **intermediate space**.[3]

Here and in what follows, we use the following terminology:

(a) the *reference space* is the ambient space for the reference body, with vectors in that space referred to as *material vectors.*

(b) the *structural space* as the ambient space for the microscopic structure, with vectors in that space referred to as *structural vectors*;

(c) the *observed space* as the ambient space for the deformed body, with vectors in that space referred to as *spatial vectors.*

Thus $\mathbf{F}^p$ and $\mathbf{F}^e$ have the following mapping properties (Figure 1):

(P1) $\mathbf{F}^p$ *maps material vectors to structural vectors*;

(P2) $\mathbf{F}^e$ *maps structural vectors to spatial vectors.*

Further, we refer to a tensor field $\mathbf{G}$ as a *spatial tensor field* if $\mathbf{G}$ maps spatial vectors to spatial vectors, a *material tensor field* if $\mathbf{G}$ maps material vectors to material vectors; in the same vein, we now refer to $\mathbf{G}$ as an **structural tensor field** if $\mathbf{G}$ *maps structural vectors to structural vectors.*

Elastic and plastic stretching and spin

The velocity gradient

$$\mathbf{L} = \operatorname{grad}\dot{\chi}$$

is related to the deformation gradient $\mathbf{F}$ through the identity

$$\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1},$$

and we may use the Kröner–Lee decomposition (2.1) to relate $\mathbf{L}$ to $\mathbf{F}^p$ and $\mathbf{F}^e$. By (2.1),

$$\dot{\mathbf{F}} = \dot{\mathbf{F}}^e\mathbf{F}^p + \mathbf{F}^p\dot{\mathbf{F}}^p, \qquad \mathbf{F}^{-1} = \mathbf{F}^{p-1}\mathbf{F}^{e-1}, \tag{2.6}$$

[2] We do not mean to infer from this that there is a vector $\mathbf{l}$ with differential $\mathbf{dl}$.
[3] Generally referred to as the relaxed or intermediate *configuration.*

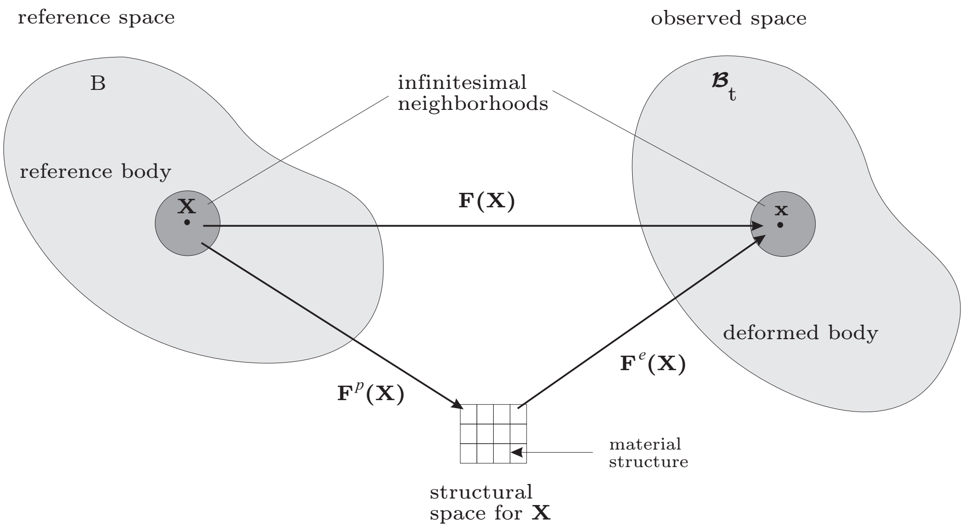

Figure 1: Schematic of the Kröner–Lee decomposition. The dark grey circles denote infinitesimal neighborhoods of the points $\mathbf{X}$ and $\mathbf{x} = \chi_t(\mathbf{X})$. The arrows are meant to indicate the mapping properties of the linear transformations $\mathbf{F}$, $\mathbf{F}^p$ and $\mathbf{F}^e$.

and therefore

$$\begin{aligned}\mathbf{L} &= (\dot{\mathbf{F}}^e\mathbf{F}^p + \mathbf{F}^e\dot{\mathbf{F}}^p)(\mathbf{F}^{p-1}\mathbf{F}^{e-1})\\ &= \dot{\mathbf{F}}^e\mathbf{F}^{e-1} + \mathbf{F}^e(\dot{\mathbf{F}}^p\mathbf{F}^{p-1})\mathbf{F}^{e-1}.\end{aligned}$$

Thus, defining **elastic** and **plastic distortion-rate tensors** $\mathbf{L}^e$ and $\mathbf{L}^p$ through the relations

$$\mathbf{L}^e = \dot{\mathbf{F}}^e\mathbf{F}^{e-1} \qquad \text{and} \qquad \mathbf{L}^p = \dot{\mathbf{F}}^p\mathbf{F}^{p-1}, \tag{2.7}$$

we have the decomposition

$$\mathbf{L} = \mathbf{L}^e + \mathbf{F}^e\mathbf{L}^p\mathbf{F}^{e-1}. \tag{2.8}$$

We define the **elastic stretching** $\mathbf{D}^e$ and the **elastic spin** $\mathbf{W}^e$ through the relations

$$\left.\begin{aligned}\mathbf{D}^e &= \tfrac{1}{2}(\mathbf{L}^e + \mathbf{L}^{e\top}),\\ \mathbf{W}^e &= \tfrac{1}{2}(\mathbf{L}^e - \mathbf{L}^{e\top}).\end{aligned}\right\} \tag{2.9}$$

Similarly, we define the **plastic stretching** $\mathbf{D}^p$ and the **plastic spin** $\mathbf{W}^p$ through

$$\left.\begin{aligned}\mathbf{D}^p &= \tfrac{1}{2}(\mathbf{L}^p + \mathbf{L}^{p\top}),\\ \mathbf{W}^p &= \tfrac{1}{2}(\mathbf{L}^p - \mathbf{L}^{p\top}).\end{aligned}\right\} \tag{2.10}$$

A general experimental observation is that at the microstructural scale, *plastic flow by dislocation motion induces negligibly small changes in volume.*[4] Consistent with this, we assume that at the macroscopic scale plastic flow does not induce a change in volume, and accordingly we assume that

$$\det \mathbf{F}^p \equiv 1. \tag{2.11}$$

Then, since

$$\overline{\det \mathbf{F}^p}^{\,\boldsymbol{\cdot}} = (\det \mathbf{F}^p)(\operatorname{tr}\mathbf{L}^p) = (\det \mathbf{F}^p)(\operatorname{tr}\mathbf{D}^p) = 0,$$

[4]Cf., e.g. Bridgeman (1944), Spitzig and Richmond (1976).

we find that $\mathbf{L}^p$ and (hence) $\mathbf{D}^p$ are *deviatoric*,

$$\operatorname{tr}\mathbf{L}^p = \operatorname{tr}\mathbf{D}^p = 0. \tag{2.12}$$

Also, since $J = \det \mathbf{F} = \det \mathbf{F}^e \det \mathbf{F}^p$, it follows that

$$J = \det \mathbf{F} = \det \mathbf{F}^e = J^e, \tag{2.13}$$

and hence that

$$\dot{J} = J \operatorname{tr}\mathbf{D}^e. \tag{2.14}$$

Consequences of (P1) and (P2) on page 2, (2.7), (2.9), and (2.10) are that

(P3) $\mathbf{L}^e$ *and* $\mathbf{D}^e$ *are spatial tensor fields*;

(P4) $\mathbf{L}^p$, $\mathbf{D}^p$, *and* $\mathbf{W}^p$ *are structural tensor fields.*

Elastic and plastic polar decompositions

As in the standard polar decompositions

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}$$

of the deformation gradient $\mathbf{F}$ into stretch and rotation tensors, our definition of the elastic and plastic stretch and rotation tensors is based on the right and left polar decompositions:

$$\left.\begin{aligned} \mathbf{F}^e &= \mathbf{R}^e\mathbf{U}^e = \mathbf{V}^e\mathbf{R}^e, \\ \mathbf{F}^p &= \mathbf{R}^p\mathbf{U}^p = \mathbf{V}^p\mathbf{R}^p. \end{aligned}\right\} \tag{2.15}$$

Here $\mathbf{R}^e$ and $\mathbf{R}^p$ are the **elastic** and **plastic rotations**, $\mathbf{U}^e$ and $\mathbf{V}^e$ are the right and left **elastic stretch tensors**, and $\mathbf{U}^p$ and $\mathbf{V}^p$ are the right and left **plastic stretch tensors**,[5] so that,

$$\left.\begin{aligned} \mathbf{U}^e &= \sqrt{\mathbf{F}^{e\top}\mathbf{F}^e}, \\ \mathbf{V}^e &= \sqrt{\mathbf{F}^e\mathbf{F}^{e\top}}, \end{aligned}\right\} \tag{2.16}$$

and

$$\left.\begin{aligned} \mathbf{U}^p &= \sqrt{\mathbf{F}^{p\top}\mathbf{F}^p}, \\ \mathbf{V}^p &= \sqrt{\mathbf{F}^p\mathbf{F}^{p\top}}. \end{aligned}\right\} \tag{2.17}$$

Further, the right and left **elastic Cauchy–Green tensors** are defined by

$$\left.\begin{aligned} \mathbf{C}^e &= \mathbf{U}^{e2} = \mathbf{F}^{e\top}\mathbf{F}^e, \\ \mathbf{B}^e &= \mathbf{V}^{e2} = \mathbf{F}^e\mathbf{F}^{e\top}, \end{aligned}\right\} \tag{2.18}$$

and the right and left **plastic Cauchy–Green tensors** tensors by

$$\left.\begin{aligned} \mathbf{C}^p &= \mathbf{U}^{p2} = \mathbf{F}^{p\top}\mathbf{F}^p, \\ \mathbf{B}^p &= \mathbf{V}^{p2} = \mathbf{F}^p\mathbf{F}^{p\top}. \end{aligned}\right\} \tag{2.19}$$

Differentiating $(2.18)_1$ results in the following expression for the rate of change of $\mathbf{C}^e$:

$$\begin{aligned} \dot{\mathbf{C}}^e &= \big(\mathbf{F}^{e\top}\dot{\mathbf{F}}^e + \dot{\mathbf{F}}^{e\top}\mathbf{F}^e\big) \\ &= \mathbf{F}^{e\top}\big(\dot{\mathbf{F}}^e\mathbf{F}^{e-1} + \mathbf{F}^{e-\top}\dot{\mathbf{F}}^{e\top}\big)\mathbf{F}^e \\ &= 2\,\mathbf{F}^{e\top}\mathbf{D}^e\mathbf{F}^e. \end{aligned} \tag{2.20}$$

Hence

$$\mathbf{D}^e = \tfrac{1}{2}\mathbf{F}^{e-\top}\dot{\mathbf{C}}^e\mathbf{F}^{e-1}. \tag{2.21}$$

Consequences of (P2) on page 2, $(2.15)_1$, and (2.18) are that:

[5] The stretch tensors are therefore symmetric, positive-definite.

(P5) $\mathbf{U}^e$ *and* $\mathbf{C}^e$ *are structural tensor fields*;

(P6) $\mathbf{R}^e$ *maps structural vectors to spatial vectors.*

2.1 Basic laws

Momentum balances. Free energy imbalance

We assume that the underlying frame is *inertial* and begin with the local **momentum balances**

$$\left.\begin{aligned} \rho\dot{\mathbf{v}} &= \mathrm{div}\mathbf{T} + \mathbf{b}_0, \\ \mathbf{T} &= \mathbf{T}^{\top}, \end{aligned}\right\} \tag{2.22}$$

in terms of the Cauchy stress $\mathbf{T}$. Also, we take as our basic thermodynamical law the **free-energy imbalance** under **isothermal condition**:

$$\rho\,\dot{\psi} - \mathbf{T}\colon\mathbf{L} \le 0 \tag{2.23}$$

with ψ *the free-energy measured per unit mass of the body*, and ρ the mass density in the deformed body.

In what follows it is helpful to express the free-energy imbalance (2.23) in terms of *the free energy and the stress-power measured per unit volume of the intermediate space.* Thus, with $\rho_{\rm I}$ the mass density in the intermediate space, let

$$\psi_{\rm I} \stackrel{\rm def}{=} \rho_{\rm I}\,\psi \tag{2.24}$$

denote *the free energy measured per unit volume of the intermediate space.* For a plastically incompressible material the mass density $\rho_{\rm I}$ is related to the mass densities $\rho_{\rm R}$ and ρ in the reference and deformed bodies by

$$\rho_{\rm R} = \rho_{\rm I}, \qquad \rho_{\rm I} = \rho\,J^e, \tag{2.25}$$

respectively. Thus, from $(2.25)_1$,

$$\dot{\rho}_{\rm I} = 0, \tag{2.26}$$

and hence

$$\rho_{\rm I}\,\dot{\psi} = \dot{\psi}_{\rm I}. \tag{2.27}$$

Next, dividing (2.23) through by ρ, multiplying the resulting expression by $\rho_{\rm I}$, and using $(2.25)_2$, we obtain

$$\rho_{\rm I}\dot{\psi} - J^e\,\mathbf{T}\colon\mathbf{L} \le 0.$$

Finally, using (2.27) we obtain

$$\delta_{\rm I} = -\dot{\psi}_{\rm I} + J^e\,\mathbf{T}\colon\mathbf{L} \ge 0, \tag{2.28}$$

where $\delta_{\rm I}$ is the dissipation rate per unit volume of the intermediate space.

Consider next the expression $J^e\,\mathbf{T}\colon\mathbf{L}$ for the stress power per unit volume of the intermediate space. Using (2.8) and the symmetry of $\mathbf{T}$ we obtain[6]

$$J^e\,\mathbf{T}\colon\mathbf{L} = \underbrace{J^e\,\mathbf{T}\colon\mathbf{D}^e}_{\text{elastic term}} + \underbrace{J^e\,(\mathbf{F}^{e\top}\mathbf{T}\mathbf{F}^{e-\top})\colon\mathbf{L}^p}_{\text{plastic term}}. \tag{2.29}$$

Based on our treatment of elastic solids we seek to express the "elastic term" $J^e\,\mathbf{T}\colon\mathbf{D}^e$ in terms of $\dot{\mathbf{C}}^e$. Using (2.21)

$$\begin{aligned} J^e\,\mathbf{T}\colon\mathbf{D}^e &= \tfrac{1}{2}J^e\mathbf{T}\colon(\mathbf{F}^{e-\top}\dot{\mathbf{C}}^e\mathbf{F}^{e-1}), \\ &= \tfrac{1}{2}\Big(J^e\mathbf{F}^{e-1}\mathbf{T}\mathbf{F}^{e-\top}\Big)\colon\dot{\mathbf{C}}^e. \end{aligned} \tag{2.30}$$

[6]We make frequent use of the identity

$$\mathbf{A}\colon(\mathbf{B}\mathbf{C}) = (\mathbf{B}^{\top}\mathbf{A})\colon\mathbf{C} = (\mathbf{A}\mathbf{C}^{\top})\colon\mathbf{B}.$$

Let

$$\mathbf{T}^e \stackrel{\text{def}}{=} J^e \mathbf{F}^{e-1}\mathbf{T}\mathbf{F}^{e-\top}. \tag{2.31}$$

This *symmetric* stress-measure is an analog of the second Piola stress with respect to the intermediate or structural space; that is, $\mathbf{T}^e$ is the second Piola stress computed using $\mathbf{F}^e$ in place of $\mathbf{F}$. Using (2.31) we may express (2.30) as

$$J^e\,\mathbf{T}\!:\!\mathbf{D}^e = \tfrac{1}{2}\mathbf{T}^e\!:\!\dot{\mathbf{C}}^e. \tag{2.32}$$

We turn our attention next to the "plastic term" in (2.29):

$$\begin{aligned} J^e\,(\mathbf{F}^{e\top}\mathbf{T}\mathbf{F}^{e-\top})\!:\!\mathbf{L}^p &= (\underbrace{\mathbf{F}^{e\top}\mathbf{F}^e}_{\mathbf{C}^e}\,\underbrace{J^e\,\mathbf{F}^{e-1}\mathbf{T}\mathbf{F}^{e-\top}}_{\mathbf{T}^e})\!:\!\mathbf{L}^p \\ &= (\mathbf{C}^e\mathbf{T}^e)\!:\!\mathbf{L}^p. \end{aligned}$$

Central to the theory is the **Mandel stress** defined by

$$\mathbf{M}^e \stackrel{\text{def}}{=} \mathbf{C}^e\mathbf{T}^e. \tag{2.33}$$

Using (2.33) in the expression above we obtain

$$J^e\,(\mathbf{F}^{e\top}\mathbf{T}\mathbf{F}^{e-\top})\!:\!\mathbf{L}^p = \mathbf{M}^e\!:\!\mathbf{L}^p. \tag{2.34}$$

Substituting (2.32) and (2.34) into (2.29), we find the following expression for the *stress power*:

$$J^e\mathbf{T}\!:\!\mathrm{L} = \underbrace{\tfrac{1}{2}\mathbf{T}^e\!:\!\dot{\mathbf{C}}^e}_{\text{elastic term}} + \underbrace{\mathbf{M}^e\!:\!\mathbf{L}^p}_{\text{plastic term}}\;. \tag{2.35}$$

Finally, using (2.35) we can write the free-energy balance (2.28) in a form suitable to a discussion of plasticity:

$$\delta_{\mathrm{I}} = -\dot{\psi}_{\mathrm{I}} + \tfrac{1}{2}\mathbf{T}^e\!:\!\dot{\mathbf{C}}^e + \mathbf{M}_0^e\!:\!\mathbf{L}^p \geq 0, \tag{2.36}$$

where in writing the plastic power term we have used the fact that $\operatorname{tr}\mathbf{L}^p = 0$.

For future use, we list together the relations (2.31) and (2.33) for $\mathbf{T}^e$ and $\mathbf{M}^e$:

$$\mathbf{T}^e = J^e\mathbf{F}^{e-1}\mathbf{T}\mathbf{F}^{e-\top} \qquad \text{and} \qquad \mathbf{M}^e = \mathbf{C}^e\mathbf{T}^e. \tag{2.37}$$

Note that $(2.37)_1$ may be inverted to give an expression for $\mathbf{T}$ as a function of $\mathbf{T}^e$:

$$\mathbf{T} = J^{-1}\mathbf{F}^e\mathbf{T}^e\mathbf{F}^{e\top}. \tag{2.38}$$

Further,

$$\begin{aligned} \mathbf{T} &= J^{e\,-1}\mathbf{F}^e\mathbf{C}^{e-1}\mathbf{M}^e\mathbf{F}^{e\top} && \text{by (2.38), (2.33)} \\ &= J^{e\,-1}\mathbf{F}^e\mathbf{F}^{e-1}\mathbf{F}^{e-\top}\mathbf{M}^e\mathbf{F}^{e\top} && \text{by (2.18)} \\ &= J^{e\,-1}\mathbf{F}^e\mathbf{M}^{e\top}\mathbf{F}^{e-1} && \text{since } \mathbf{T} \text{ is symmetric;} \end{aligned}$$

the Cauchy and Mandel stresses are therefore related by

$$\mathbf{T} = J^{e\,-1}\mathbf{F}^e\mathbf{M}^{e\top}\mathbf{F}^{e-1}. \tag{2.39}$$

Our next step is to determine the mapping properties of $\mathbf{T}^e$, and $\mathbf{M}^e$. As is clear from (2.31), the input space for $\mathbf{T}^e$ is the same as that for $\mathbf{F}^{e-\top}$, which, by (P2) on page 2, is the structural space. Similarly, the output space for $\mathbf{T}^e$ is the same as that for $\mathbf{F}^{e-1}$, which is again the structural space. Thus $\mathbf{T}^e$ maps structural vectors to structural vectors. Further, by (P5) on page 5, $\mathbf{C}^e$ also has this mapping property, and we therefore may conclude from $(2.37)_2$ that $\mathbf{M}^e$ also map structural vectors to structural vectors. Thus:

(P7) $\mathbf{T}^e$ *and* $\mathbf{M}^e$ *are structural tensor fields.*

2.2 Frame-indifference

Recall that a **change of frame** at each *fixed time* t, defined by a rotation $\mathbf{Q}(t)$ and a vector $\mathbf{y}(t)$ and transforms *spatial points* $\mathbf{x}$ to *spatial points*

$$\mathbf{x}^* = \mathbf{y}(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{o}). \tag{2.40}$$

As discussed earlier, a change of frame is, at each time, a rotation and translation of the *observed space* (the space through which the body moves); *it does not affect the reference space, nor does it affect the structural space*; thus

(‡) *material vectors and structural vectors are invariant under changes in frame*,

an assertion that should be at least intuitively clear from Figure 1.

Since observers view only the deformed body, tensor fields that map material vectors to material vectors are invariant under changes in frame. In view of (‡), the exact same argument yields the following result:

(†) *tensor fields*

(a) *that map material vectors to material vectors, or*

(b) *that map material vectors to structural vectors, or*

(c) *that map structural vectors to material vectors, or*

(d) *that map structural vectors to structural vectors*,

are invariant under changes in frame.

Next, recall the transformation law for the deformation gradient $\mathbf{F}$ under a change in frame:

$$\mathbf{F}^* = \mathbf{Q}\mathbf{F}. \tag{2.41}$$

By (2.1) and (2.41),

$$(\mathbf{F}^e\mathbf{F}^p)^* = \mathbf{Q}(\mathbf{F}^e\mathbf{F}^p).$$

On the other hand, by (P2) and (b) of (†), $(\mathbf{F}^p)^* = \mathbf{F}^p$, so that

$$\begin{aligned}(\mathbf{F}^e\mathbf{F}^p)^* &= (\mathbf{F}^e)^*(\mathbf{F}^p)^* \\ &= (\mathbf{F}^e)^*\mathbf{F}^p;\end{aligned}$$

hence

$$\mathbf{Q}\mathbf{F}^e\mathbf{F}^p = (\mathbf{F}^e)^*\mathbf{F}^p.$$

Thus[7]

$$(\mathbf{F}^e)^* = \mathbf{Q}\mathbf{F}^e, \qquad \text{and} \qquad \mathbf{F}^p \text{ is invariant.} \tag{2.42}$$

Similarly, appealing to (P4) and (b) of (†),

$$\mathbf{L}^p \text{ is invariant.} \tag{2.43}$$

Next, by (2.15) and (2.42),

$$\begin{aligned}(\mathbf{F}^e)^* = (\mathbf{R}^e)^*(\mathbf{U}^e)^* = \mathbf{Q}\mathbf{F}^e = \underline{\mathbf{Q}\mathbf{R}^e}\,\mathbf{U}^e, \\ (\mathbf{F}^e)^* = (\mathbf{V}^e)^*(\mathbf{R}^e)^* = \mathbf{Q}\mathbf{F}^e = \underline{\mathbf{Q}\mathbf{V}^e\mathbf{Q}^\top}\,\mathbf{Q}\mathbf{R}^e,\end{aligned}$$

and we may conclude from the uniqueness of the polar decomposition that $(\mathbf{R}^e)^* = \mathbf{Q}\mathbf{R}^e$ and

$$\mathbf{U}^e \text{ is invariant}, \qquad (\mathbf{V}^e)^* = \mathbf{Q}\mathbf{V}^e\mathbf{Q}^\top, \tag{2.44}$$

[7] Silahavy (1977).

so that, by (2.18),

$$\mathbf{C}^e \text{ is invariant}, \qquad (\mathbf{B}^e)^* = \mathbf{Q}\mathbf{B}^e\mathbf{Q}^\top. \tag{2.45}$$

We next establish transformation rules for stresses $\mathbf{T}^e$ and $\mathbf{M}^e$ expressed, via (2.37), as functions of the Cauchy stress $\mathbf{T}$, which is frame-indifferent. By $(2.37)_1$ and (2.41),

$$\begin{aligned}
(\mathbf{T}^e)^* &= J^e\,[(\mathbf{F}^e)^*]^{-1}\mathbf{T}^*\,[(\mathbf{F}^e)^*]^{-\top}\\
&= J^e\,[\mathbf{Q}\mathbf{F}^e]^{-1}\mathbf{Q}\mathbf{T}\mathbf{Q}^\top\,[(\mathbf{Q}\mathbf{F}^e)]^{-\top}\\
&= J^e\,[\mathbf{F}^{e-1}\mathbf{Q}^\top\mathbf{Q}\mathbf{T}\mathbf{Q}^\top\mathbf{Q}\mathbf{F}^{e-\top}\\
&= J^e\,\mathbf{F}^{e-1}\mathbf{T}\mathbf{F}^{e-\top}\\
&= \mathbf{T}^e.
\end{aligned} \tag{2.46}$$

Finally, (2.33), (2.45), and (2.46) imply that $(\mathbf{M}^e)^* = \mathbf{M}^e$. Thus

$$\mathbf{T}^e, \text{ and } \mathbf{M}^e \text{ are invariant.} \tag{2.47}$$

3 Constitutive theory

We neglect defect energy and restrict attention to constitutive relations that separate elastic and plastic constitutive response. Therefore, guided by (2.36), we consider elastic constitutive relations of the form

$$\left.\begin{aligned}
\psi_{\text{I}} &= \bar{\psi}_{\text{I}}(\mathbf{C}^e, \vartheta) \quad \text{with} \quad \bar{\psi}_{\text{I}}(\mathbf{1}, \vartheta) = 0,\\
\mathbf{T}^e &= \bar{\mathbf{T}}^e(\mathbf{C}^e, \vartheta).
\end{aligned}\right\} \tag{3.1}$$

Recall, that we are focussing on an *isothermal theory*, and hence the temperature ϑ is *constant* of the theory. For the plastic constitutive equations we introduce a list of n scalar *internal variables* $\vec{S} = (S_1, S_2, \ldots, S_n)$, and assume that

$$\left.\begin{aligned}
&\mathbf{L}^p = \bar{\mathbf{L}}^p(\mathbf{M}^e, \vec{S}, \vartheta), \quad \text{with} \quad \operatorname{tr}\bar{\mathbf{L}}^p(\mathbf{M}^e, \vec{S}, \vartheta) = 0 \qquad \text{and} \qquad \bar{\mathbf{L}}^p(\mathbf{0}, \vec{S}, \vartheta) = \mathbf{0}\\
&\dot{S}_i = H_i(\mathbf{L}^p, \vec{S}, \vartheta).
\end{aligned}\right\} \tag{3.2}$$

Note that by (2.43) and $(2.45)_1$, $\mathbf{C}^e$ and $\mathbf{L}^p$ are invariant under changes in frame, and by (2.47) so also are $\mathbf{T}^e$ and $\mathbf{M}^e$. Thus, since $\vec{S}$ — being a scalar fields — are frame-indifferent, *the constitutive equations* (3.1) *and* (3.2) *are frame-indifferent.*

Under isothermal conditions,

$$\dot{\psi}_{\text{I}} = \frac{\partial\bar{\psi}_{\text{I}}(\mathbf{C}^e, \vartheta)}{\partial\mathbf{C}^e} : \dot{\mathbf{C}}^e, \tag{3.3}$$

and hence satisfaction of the free-energy imbalance (2.36) requires that

$$\left[\tfrac{1}{2}\bar{\mathbf{T}}^e(\mathbf{C}^e, \vartheta) - \frac{\partial\bar{\psi}_{\text{I}}(\mathbf{C}^e, \vartheta)}{\partial\mathbf{C}^e}\right] : \dot{\mathbf{C}}^e + \mathbf{M}^e_0 : \bar{\mathbf{L}}^p(\mathbf{M}^e, \vec{S}, \vartheta) \geq 0, \tag{3.4}$$

hold in all motions of the body. Thus, *sufficient* conditions that the constitutive equations satisfy the free-energy imbalance are that

(i) *the free energy determine the stress through the* **stress relation**

$$\bar{\mathbf{T}}^e(\mathbf{C}^e, \vartheta) = 2\frac{\partial\bar{\psi}_{\text{I}}(\mathbf{C}^e, \vartheta)}{\partial\mathbf{C}^e}; \tag{3.5}$$

(ii) *the plastic distortion-rate* $\mathbf{L}^p$ *satisfy the* **reduced dissipation inequality**

$$\mathbf{M}_0^e : \bar{\mathbf{L}}^p(\mathbf{M}^e, \vec{S}, \vartheta) \geq 0 \tag{3.6}$$

for all $\mathbf{M}^e$ and all $\vec{S}$.

We assume henceforth that (3.5) holds in all motions of the body, and that the material is **strictly dissipative** in the sense that

$$\mathbf{M}_0^e : \bar{\mathbf{L}}^p(\mathbf{M}^e, \vec{S}, \vartheta) > 0 \qquad \text{whenever} \qquad \mathbf{L}^p \neq \mathbf{0}. \tag{3.7}$$

4 Isotropy

Consistent with Figure 1, (P2) on page 2, a standard discussion of material symmetry for elastic solids may be applied to the constitutive equations (3.1) and (3.2) provided the roles of the *intermediate space* and the elastic distortion $\mathbf{F}^e$ in the present discussion play the roles of the *reference space* and the deformation gradient $\mathbf{F}$ in standard discussions of symmetry considerations for elastic solids.[8]

The following definitions help to make precise our notion of an isotropic material:

(i) $\mathrm{Orth}^+ =$ the group of all rotations (the proper orthogonal group);

(ii) the *symmetry group* $\mathcal{G}_{\mathrm{I}}$ at each time t, is the group of all rotations of the *intermediate* structural space that leaves the response of the material unaltered.

Let $\mathbf{Q}$, a *time-independent rotation of the intermediate space*, be a symmetry transformation. Then $\mathbf{F}$ is unaltered by such a rotation, and hence[9]

$$\mathbf{F}^e \to \mathbf{F}^e\mathbf{Q} \quad \text{and} \quad \mathbf{F}^p \to \mathbf{Q}^{\top}\mathbf{F}^p, \tag{4.1}$$

and also

$$\mathbf{C}^e \to \mathbf{Q}^{\top}\mathbf{C}^e\mathbf{Q}, \qquad \dot{\mathbf{C}}^e \to \mathbf{Q}^{\top}\dot{\mathbf{C}}^e\mathbf{Q}, \qquad \mathbf{L}^p \to \mathbf{Q}^{\top}\mathbf{L}^p\mathbf{Q}. \tag{4.2}$$

To deduce the transformation laws for $\mathbf{T}^e$ and $\mathbf{M}^e$ under a symmetry transformation,

- *we require invariance of the internal power (2.35) under a symmetry transformation; i.e., we require that*

$$\mathbf{T}^e : \dot{\mathbf{C}}^e \quad \textit{and} \quad \mathbf{M}^e : \mathbf{L}^p \quad \textit{be invariant} \tag{4.3}$$

  under a symmetry transformation $\mathbf{Q}$. Then (4.2) and (4.3) yield the transformation laws

$$\mathbf{T}^e \to \mathbf{Q}^{\top}\mathbf{T}^e\mathbf{Q}, \qquad \mathbf{M}^e \to \mathbf{Q}^{\top}\mathbf{M}^e\mathbf{Q}. \tag{4.4}$$

Thus we conclude that

$$\left.\begin{aligned}
\bar{\psi}_{\mathrm{I}}(\mathbf{C}^e, \vartheta) &= \bar{\psi}_{\mathrm{I}}(\mathbf{Q}^{\top}\mathbf{C}^e\mathbf{Q}, \vartheta),\\
\mathbf{Q}^{\top}\bar{\mathbf{T}}^e(\mathbf{C}^e, \vartheta)\mathbf{Q} &= \bar{\mathbf{T}}^e(\mathbf{Q}^{\top}\mathbf{C}^e\mathbf{Q}, \vartheta),\\
\mathbf{Q}^{\top}\bar{\mathbf{L}}^p(\mathbf{M}^e, \vec{S}, \vartheta)\mathbf{Q} &= \bar{\mathbf{L}}^p(\mathbf{Q}^{\top}\mathbf{M}^e\mathbf{Q}, \vec{S}, \vartheta),\\
H_i(\mathbf{L}^p, \vec{S}, \vartheta) &= H_i(\mathbf{Q}^{\top}\mathbf{L}^p\mathbf{Q}, \vec{S}, \vartheta),
\end{aligned}\right\} \tag{4.5}$$

must hold for all rotations $\mathbf{Q}$ in the *symmetry group* $\mathcal{G}_{\mathrm{I}}$ at each time t.

We refer to the material as *isotropic* (and to the intermediate space as undistorted) if at each time t

$$\mathcal{G}_{\mathrm{I}} = \mathrm{Orth}^+, \tag{4.6}$$

so that the response of the material is invariant under arbitrary rotations of the intermediate space. We henceforth restrict attention to materials that are *isotropic*. In this case, the response functions $\bar{\psi}_{\mathrm{I}}$, $\bar{\mathbf{T}}^e$, $\bar{\mathbf{L}}^p$, and H_i must each be *isotropic*.

[8] Within the framework of the Kröner–Lee decomposition the structural space (rather than the reference space) represents the seat of material structure; cf. Figure 1.

[9] Hahn (1974), Anand (1980).

4.1 Consequences of isotropy of the elastic response

Since $\bar{\psi}_{\rm I}(\mathbf{C}^e, \vartheta)$ is an isotropic function of $\mathbf{C}^e$, it has the representation

$$\psi_{\rm I} = \tilde{\psi}_{\rm I}(\mathcal{I}_{\mathbf{C}^e}, \vartheta), \tag{4.7}$$

where

$$\mathcal{I}_{\mathbf{C}^e} = \Big(I_1(\mathbf{C}^e), I_2(\mathbf{C}^e), I_3(\mathbf{C}^e), \vartheta\Big)$$

is the list of principal invariants of $\mathbf{C}^e$. Let

$$(\lambda_1^e, \lambda_2^e, \lambda_3^e)$$

denote the positive eigenvalues of $\mathbf{U}^e$. Then the principal invariants of $\mathbf{C}^e$ may be expressed as

$$\left.\begin{aligned} I_1(\mathbf{C}^e) &= \lambda_1^{e\,2} + \lambda_2^{e\,2} + \lambda_3^{e\,2}, \\ I_2(\mathbf{C}^e) &= \lambda_1^{e\,2}\lambda_2^{e\,2} + \lambda_2^{e\,2}\lambda_3^{e\,2} + \lambda_3^{e\,2}\lambda_1^{e\,2}, \\ I_3(\mathbf{C}^e) &= \lambda_1^{e\,2}\lambda_2^{e\,2}\lambda_3^{e\,2}. \end{aligned}\right\} \tag{4.8}$$

In writing (4.8) it is tacit that the list $(\lambda_1^e, \lambda_2^e, \lambda_3^e)$ of principal stretches is presumed to have *each stretch repeated a number of times equal to its multiplicity as an eigenvalue of* $\mathbf{U}^e$. Using (4.8) in (4.7), to express the free energy in terms of the principal stretches, we obtain:

$$\begin{aligned} \psi_{\rm I} &= \tilde{\psi}_{\rm I}(\mathcal{I}_{\mathbf{C}^e}, \vartheta) \\ &= \breve{\psi}_{\rm I}(\lambda_1^e, \lambda_2^e, \lambda_3^e, \vartheta). \end{aligned} \tag{4.9}$$

Since the expressions (4.8) for $I_1(\mathbf{C}^e)$, $I_2(\mathbf{C}^e)$, and $I_3(\mathbf{C}^e)$ in terms of λ_1^e, λ_2^e, and λ_3^e are invariant under permutations of the integers $(1, 2, 3)$ labelling the principal stretches, so also is $\breve{\psi}_{\rm I}(\lambda_1^e, \lambda_2^e, \lambda_3^e)$; i.e.,

$$\breve{\psi}_{\rm I}(\lambda_1^e, \lambda_2^e, \lambda_3^e, \vartheta) = \breve{\psi}_{\rm I}(\lambda_1^e, \lambda_3^e, \lambda_2^e, \vartheta) \quad \text{and so forth.}$$

Next, let

$$\omega_i^e = \lambda_i^{e\,2}, \quad i = 1, 2, 3. \tag{4.10}$$

Then, by the chain-rule and (3.5), the stress $\mathbf{T}^e$ is given by

$$\begin{aligned} \mathbf{T}^e &= 2\,\frac{\partial \breve{\psi}_{\rm I}(\lambda_1^e, \lambda_2^e, \lambda_3^e, \vartheta)}{\partial \mathbf{C}^e} \\ &= 2\sum_{i=1}^{3} \frac{\partial \breve{\psi}_{\rm I}(\lambda_1^e, \lambda_2^e, \lambda_3^e, \vartheta)}{\partial \lambda_i^e}\,\frac{\partial \lambda_i^e}{\partial \mathbf{C}^e} \\ &= \sum_{i=1}^{3} \frac{1}{\lambda_i^e}\,\frac{\partial \breve{\psi}_{\rm I}(\lambda_1^e, \lambda_2^e, \lambda_3^e, \vartheta)}{\partial \lambda_i^e}\,\frac{\partial \omega_i}{\partial \mathbf{C}^e}. \end{aligned} \tag{4.11}$$

The spectral representation of $\mathbf{C}^e$ is

$$\mathbf{C}^e = \sum_{i=1}^{3} \omega_i^e \mathbf{r}_i^e \otimes \mathbf{r}_i^e, \qquad \omega_i^e = \lambda_i^{e\,2}, \tag{4.12}$$

where $(\mathbf{r}_1^e, \mathbf{r}_2^e, \mathbf{r}_2^e)$ are the orthonormal eigenvectors of $\mathbf{C}^e$ (and $\mathbf{U}^e$). Assume that the squared principal stretches ω_i^e are distinct, so that the ω_i^e and the principal directions $\mathbf{r}_i^e$ may be considered as functions of $\mathbf{C}^e$. Then,

$$\frac{\partial \omega_i^e}{\partial \mathbf{C}^e} = \mathbf{r}_i^e \otimes \mathbf{r}_i^e, \tag{4.13}$$

and, granted this, (4.13) and (4.11) imply that

$$\mathbf{T}^e = \sum_{i=1}^{3} \frac{1}{\lambda_i^e} \frac{\partial \breve{\psi}_{\rm I}(\lambda_1^e, \lambda_2^e, \lambda_3^e, \vartheta)}{\partial \lambda_i^e} \, \mathbf{r}_i^e \otimes \mathbf{r}_i^e. \tag{4.14}$$

Next, since

$$\mathbf{F}^e = \sum_{i=1}^{3} \lambda_i^e \, \mathbf{l}_i^e \otimes \mathbf{r}_i^e \tag{4.15}$$

where

$$\mathbf{l}_i^e = \mathbf{R}^e \, \mathbf{r}_i^e,$$

are the eigenvectors of $\mathbf{V}^e$ (or $\mathbf{B}^e$), use of (2.38) and (4.14) gives

$$\mathbf{T} = J^{e\,-1} \Big(\sum_{i=1}^{3} \lambda_i^e \, \mathbf{l}_i^e \otimes \mathbf{r}_i^e \Big) \Big(\sum_{i=1}^{3} \frac{1}{\lambda_i^e} \frac{\partial \breve{\psi}_{\rm I}(\lambda_1^e, \lambda_2^e, \lambda_3^e)}{\partial \lambda_i^e} \, \mathbf{r}_i^e \otimes \mathbf{r}_i^e \Big) \Big(\sum_{i=1}^{3} \lambda_i^e \, \mathbf{r}_i^e \otimes \mathbf{l}_i^e \Big),$$

or

$$\mathbf{T} = J^{e\,-1} \sum_{i=1}^{3} \lambda_i^e \frac{\partial \breve{\psi}_{\rm I}(\lambda_1^e, \lambda_2^e, \lambda_3^e, \vartheta)}{\partial \lambda_i^e} \, \mathbf{l}_i^e \otimes \mathbf{l}_i^e. \tag{4.16}$$

Also, use of (4.12) and (4.14) in $(2.37)_2$ gives

$$\mathbf{M}^e = \sum_{i=1}^{3} \lambda_i^e \frac{\partial \breve{\psi}_{\rm I}(\lambda_1^e, \lambda_2^e, \lambda_3^e, \vartheta)}{\partial \lambda_i^e} \, \mathbf{r}_i^e \otimes \mathbf{r}_i^e. \tag{4.17}$$

Further, (4.16) and (4.17) yield the important relation

$$\mathbf{M}^e = J^e \, \mathbf{R}^{e\top} \mathbf{T} \mathbf{R}^e, \tag{4.18}$$

and hence that

- *the Mandel stress* $\mathbf{M}^e$ *for isotropic materials is* **symmetric**.

4.1.1 Specialization of the elastic energy

Let,

$$E_i^e \overset{\rm def}{=} \ln \lambda_i^e \tag{4.19}$$

define *principal elastic logarithmic strains*, and consider a free energy function of the form

$$\breve{\psi}_{\rm I}(\lambda_1^e, \lambda_2^e, \lambda_3^e, \vartheta) = \hat{\psi}_{\rm I}(E_1^e, E_2^e, E_3^e), \tag{4.20}$$

so that, using (4.17)

$$\mathbf{M}^e = \sum_{i=1}^{3} \frac{\partial \hat{\psi}_{\rm I}(E_1^e, E_2^e, E_3^e)}{\partial E_i^e} \, \mathbf{r}_i^e \otimes \mathbf{r}_i^e. \tag{4.21}$$

In metallic materials the elastic strains are in general "small." Accordingly, we consider the following simple generalization of the classical strain energy function of infinitesimal isotropic elasticity which uses a logarithmic measure of finite strain, and is a useful free energy function for *moderatley large elastic stretches*,[10]

$$\hat{\psi}_{\rm I}(E_1^e, E_2^e, E_3^e, \vartheta) = G \, \left[(E_1^e)^2 + (E_2^e)^2 + (E_3^e)^2 \right] + \tfrac{1}{2} \, (K - \tfrac{2}{3} G) \, \left(E_1^e + E_2^e + E_3^e \right)^2, \tag{4.22}$$

where the parameters

$$G(\vartheta) > 0, \quad \text{and} \quad K(\vartheta) > 0 \tag{4.23}$$

[10] Anand (1979, 1986)

are the shear modulus, and bulk modulus, respectively. Then, (4.21) gives

$$\mathbf{M}^e = \sum_{i=1}^{3} \Big(2GE_i^e + (K - \frac{2}{3}G)\ (E_1^e + E_2^e + E_3^e)\Big)\, \mathbf{r}_i^e \otimes \mathbf{r}_i^e. \tag{4.24}$$

Let

$$\mathbf{E}^e \stackrel{\text{def}}{=} \sum_{i=1}^{3} E_i^e\, \mathbf{r}_i^e \otimes \mathbf{r}_i^e, \tag{4.25}$$

denote the logarithmic elastic strain tensor in the intermediate space. Then, (4.24) for the Mandel stress may be written as the simple relation

$$\mathbf{M}^e = 2G\mathbf{E}_0^e + K\ (\operatorname{tr}\mathbf{E}^e)\,\mathbf{1}. \tag{4.26}$$

4.2 Consequences of isotropy of the plastic response

Let

$$\bar{\mathbf{L}}^p(\mathbf{M}^e, \vec{S}, \vartheta) = \bar{\mathbf{D}}^p(\mathbf{M}^e, \vec{S}, \vartheta) + \bar{\mathbf{W}}^p(\mathbf{M}^e, \vec{S}, \vartheta), \tag{4.27}$$

so that

$$\left.\begin{aligned} \mathbf{D}^p &= \bar{\mathbf{D}}^p(\mathbf{M}^e, \vec{S}, \vartheta), \\ \mathbf{W}^p &= \bar{\mathbf{W}}^p(\mathbf{M}^e, \vec{S}, \vartheta). \end{aligned}\right\} \tag{4.28}$$

Then, on account of $(4.5)_3$, for an isotropic material

$$\left.\begin{aligned} \mathbf{Q}^\top \bar{\mathbf{D}}^p(\mathbf{M}^e, \vec{S}, \vartheta)\mathbf{Q} &= \bar{\mathbf{D}}^p(\mathbf{Q}^\top \mathbf{M}^e \mathbf{Q}, \vec{S}, \vartheta), \\ \mathbf{Q}^\top \bar{\mathbf{W}}^p(\mathbf{M}^e, \vec{S}, \vartheta)\mathbf{Q} &= \bar{\mathbf{W}}^p(\mathbf{Q}^\top \mathbf{M}^e \mathbf{Q}, \vec{S}, \vartheta), \end{aligned}\right\} \tag{4.29}$$

must hold for all $\mathbf{Q}$. That is $\bar{\mathbf{D}}^p$ and $\bar{\mathbf{W}}^p$ are isotropic functions of the *symmetric* stress $\mathbf{M}^e$, $\vec{S}$ and ϑ. An immediate consequence of the isotropy of $\bar{\mathbf{W}}^p$, the symmetry of $\mathbf{M}^e$, and a standard representation theorem for skew tensors[11] is that

- *for isotropic materials the plastic spin vanishes,*[12]

$$\mathbf{W}^p = \mathbf{0}\,. \tag{4.30}$$

Hence the plastic constitutive equations reduce to

$$\left.\begin{aligned} \mathbf{D}^p &= \bar{\mathbf{D}}^p(\mathbf{M}^e, \vec{S}, \vartheta), \quad \text{with} \quad \operatorname{tr}\bar{\mathbf{D}}^p(\mathbf{M}^e, \vec{S}, \vartheta) = 0 \qquad \text{and} \qquad \bar{\mathbf{D}}^p(\mathbf{0}, \vec{S}, \vartheta) = \mathbf{0}, \\ \dot{S}_i &= H_i(\mathbf{D}^p, \vec{S}, \vartheta), \end{aligned}\right\} \tag{4.31}$$

with $\bar{\mathbf{D}}^p$ and H_i isotropic functions. Further, by $(2.7)_2$,

$$\dot{\mathbf{F}}^p = \mathbf{D}^p\,\mathbf{F}^p, \tag{4.32}$$

with $\mathbf{D}^p$ given by $(4.31)_1$.

Also, on account of (4.30),[13] the dissipation inequality (3.7) reduces to

$$\delta_{\rm I} = \mathbf{M}_0^e : \bar{\mathbf{D}}^p(\mathbf{M}^e, \vec{S}, \vartheta) > 0 \qquad \text{whenever} \qquad \mathbf{D}^p \neq \mathbf{0}. \tag{4.33}$$

[11] Wang (1970).
[12] Anand (1980).
[13] And also because of the symmetry of $\mathbf{M}^e$.

4.2.1 Maximum resolved stress and other hypothesis

Let

$$d^p \stackrel{\text{def}}{=} |\mathbf{D}^p|, \tag{4.34}$$

denote a **scalar flow rate**, so that the **plastic flow direction** is given by

$$\mathbf{N}^p = \frac{\mathbf{D}^p}{d^p}. \tag{4.35}$$

Then, the dissipation inequality (4.33) may be written as

$$\delta_{\text{I}} = \tau\, d^p > 0 \qquad \text{whenever} \qquad d^p > 0, \tag{4.36}$$

where

$$\tau \stackrel{\text{def}}{=} \mathbf{M}_0^e \!:\! \mathbf{N}^p > 0 \tag{4.37}$$

is a positive-valued *resolved stress* during plastic flow ($d^p > 0$).

Consider a given time t at which the Mandel stress $\mathbf{M}^e$, hardening variables $\vec{S}$ and the temperature ϑ are known, and fixed. In order to determine the direction of plastic flow $\mathbf{N}^p$ at this fixed state $(\mathbf{M}^e, \vec{S}, \vartheta)$, we make the physical assumption that for isotropic materials

- *plastic flow occurs in a direction* $\mathbf{N}^p$ *which* **maximizes** *the resolved stress* τ.[14]

To establish the consequence of this hypothesis, note that the Schwarz inequality with $|\mathbf{N}^p| = 1$ requires that

$$\tau = \mathbf{M}_0^e \!:\! \mathbf{N}^p \leq |\mathbf{M}_0^e||\mathbf{N}^p| = |\mathbf{M}_0^e|, \tag{4.38}$$

so that

$$\tau_{\max} = |\mathbf{M}_0^e|. \tag{4.39}$$

Thus using (4.37) and (4.39) we see that the hypothesis that plastic flow occurs in a direction $\mathbf{N}^p$ which maximizes the resolved stress τ leads to the important result that

- *the direction of plastic flow* $\mathbf{N}^p$ *must coincide with the direction of the deviatoric stress* $\mathbf{M}_0^e$:

$$\mathbf{N}^p = \frac{\mathbf{M}_0^e}{|\mathbf{M}_0^e|}. \tag{4.40}$$

We assume henceforth that (4.40), that is the **co-directionality** of $\mathbf{N}^p$ and $\mathbf{M}_0^e$, holds.[15]

4.2.2 Equivalent shear stress; equivalent plastic shear strain rate; equivalent plastic shear strain

We call the scalar stress measure defined by

$$\bar{\tau} \stackrel{\text{def}}{=} \sqrt{1/2}\,|\mathbf{M}_0^e|, \tag{4.42}$$

the **equivalent shear stress**. [16] Correspondingly, an **equivalent plastic shear strain rate** is defined by

$$\sqrt{2}\,|\mathbf{D}^p|;$$

[14]Note that the maximum resolved stress hypothesis is equivalent to a *maximum dissipation hypothesis.*

[15]Recall that the constitutive equation for the extra stress as a function of the stretching in an incompressible Newtonian fluid has the simple form $\mathbf{S} = 2\mu\mathbf{D}$ and hence trivially satisfies

$$\frac{\mathbf{S}}{|\mathbf{S}|} = \frac{\mathbf{D}}{|\mathbf{D}|}. \tag{4.41}$$

Equation (4.40) is the counterpart of this relation for an isotropic elastic-viscoplastic solid.

[16]Traditionally the equivalent shear stress is defined in terms of the Cauchy stress as

$$\bar{\tau} \stackrel{\text{def}}{=} \sqrt{1/2}|\mathbf{T}_0|,$$

and is so-named because when T_{12} is the only non-zero component of the Cauchy stress in shear with respect to a rectangular Cartesian coordinate system with base vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$,

$$\bar{\tau} \equiv |T_{12}|.$$

for economy of notation we replace our original definition $d^p = |\mathbf{D}^p|$ by

$$d^p \stackrel{\text{def}}{=} \sqrt{2}\,|\mathbf{D}^p|, \tag{4.43}$$

where d^p now denotes the equivalent plastic shear strain rate.[17] The quantity

$$\bar{\gamma}^p(t) \stackrel{\text{def}}{=} \int_0^t d^p(\zeta)\,d\zeta, \tag{4.44}$$

defined in terms of the equivalent plastic shear strain rate d^p, is called the **equivalent plastic shear strain** and is often used as a scalar measure of the "amount of plastic strain" at time t.

In terms of the equivalent shear stress $\bar{\tau}$ and the equivalent plastic shear strain rate d^p, the dissipation inequality may be written as

$$\delta_{\text{I}} = \bar{\tau}\,d^p > 0 \qquad \text{whenever} \qquad d^p > 0. \tag{4.45}$$

4.2.3 Hypothesis of no flow whenever the equivalent shear stress vanishes

Next, on account of the isotropy of the constitutive equation for the plastic stretching, the equivalent plastic shear strain rate is given by

$$d^p = \hat{d}^p(\mathcal{I}_{\mathbf{M}^e}, \vec{S}, \vartheta) \geq 0, \tag{4.46}$$

where

$$\mathcal{I}_{\mathbf{M}^e} = (\bar{\tau}, \bar{p}, \det \mathbf{M}^e_0) \tag{4.47}$$

is a a list of the invariants of $\mathbf{M}^e$, with

$$\bar{p} \stackrel{\text{def}}{=} -\frac{1}{3}\text{tr}\,\mathbf{M}^e, \qquad \text{and} \qquad \det \mathbf{M}^e_0 (\equiv \frac{1}{3}\text{tr}\,(\mathbf{M}^{e\,3}_0)). \tag{4.48}$$

The invariant $\bar{p}$ is called the a *mean normal pressure*.[18]

Motivated by the dissipation inequality (4.45) we make the physical hypothesis that

[17] This terminology arises from the fact that in shear, when the components of $\mathbf{D}^p$ with respect to a rectangular Cartesian system are

$$D^p_{12} = D^p_{21} \neq 0, \quad \text{and all other} \quad D^p_{ij} = 0,$$

the equivalent plastic shear strain rate is

$$d^p \equiv 2\,|D^p_{12}|.$$

[18] Two *normalized invariants* often used to describe the effects of the mean normal pressure $\bar{p}$ and the third invariant $\det \mathbf{M}^e_0$ *relative* to the equivalent shear stress are:

(i) The *stress-triaxiality parameter*,

$$X \stackrel{\text{def}}{=} \frac{-\sqrt{3}\bar{p}}{\bar{\tau}}. \tag{4.49}$$

(ii) The *Lode-angle* or *deviatoric polar angle* Θ defined by

$$\cos(3\Theta) \stackrel{\text{def}}{=} \frac{3\sqrt{3}}{2}\frac{\det \mathbf{M}^e_0}{\bar{\tau}^3}, \qquad \text{with} \qquad 0 \leq \Theta \leq \frac{\pi}{3}. \tag{4.50}$$

Writing

$$\xi \stackrel{\text{def}}{=} \cos(3\Theta), \tag{4.51}$$

and expressing $\det \mathbf{M}^e_0$ in terms of the principal values $\{\sigma_1, \sigma_2, \sigma_3\}$ of the Mandel stress $\mathbf{M}^e$ and the mean normal pressure $\bar{p}$, we note that

$$\xi = \frac{3\sqrt{3}}{2}\frac{(\sigma_1 + \bar{p})(\sigma_2 + \bar{p})(\sigma_3 + \bar{p})}{\bar{\tau}^3} \qquad \text{with} \qquad -1 \leq \xi \leq 1. \tag{4.52}$$

Note that when σ_1 is the only nonzero component of stress, $\xi = 1$ if $\sigma_1 > 0$ (simple tension), while $\xi = -1$ if $\sigma_1 < 0$ (simple compression). Also, if any $(\sigma_i + \bar{p}) = 0$, then $\xi = 0$; such a stress-state gives rise to one of the normal components of the plastic stretching to be zero, and is accordingly called a state of *plane plastic strain*.

- *no matter what the values of the invariants* $\bar{p}$ *and* $\det \mathbf{M}_0^e$*, the equivalent plastic shear strain rate vanishes whenever the equivalent shear stress vanishes*:

$$d^p = 0 \quad \text{if} \quad \bar{\tau} = 0. \tag{4.53}$$

That is

$$d^p = \hat{d}^p(\bar{\tau}, \bar{p}, \det \mathbf{M}_0^e, \vec{S}, \vartheta) \geq 0, \quad \text{with} \quad \hat{d}^p(0, \bar{p}, \det \mathbf{M}_0^e, \vec{S}, \vartheta) = 0\,. \tag{4.54}$$

4.2.4 Strong isotropy hypothesis

For an isotropic viscoplastic material we require in addition that

- the functions $H_i(d^p, \mathbf{N}^p, \vec{S}, \vartheta)$ characterizing the evolution equations for the scalar internal variables be independent of the flow direction $\mathbf{N}^p$, an assumption we refer to as the **strong isotropy hypothesis**.

4.2.5 Generalized Lévy–Mises–Reuss flow rule. Evolution equations for internal variables

In view of the hypotheses above, the constitutive equation for the plastic stretching takes the form:

$$\left.\begin{aligned} \mathbf{D}^p &= \sqrt{1/2}\, d^p\, \mathbf{N}^p, \\ \mathbf{N}^p &= \sqrt{1/2}\,(\mathbf{M}_0^e/\bar{\tau}), \\ d^p &= \hat{d}^p(\bar{\tau}, \bar{p}, \det \mathbf{M}_0^e, \vec{S}, \vartheta) \geq 0 \quad \text{with} \quad \hat{d}^p(0, \bar{p}, \det \mathbf{M}_0^e, \vec{S}, \vartheta) = 0, \end{aligned}\right\} \tag{4.55}$$

while the evolution equations for the internal variables become

$$\dot{S}_i = H_i(d^p, \vec{S}, \vartheta). \tag{4.56}$$

We refer to (4.55) as a *generalized* **Lévy–Mises–Reuss flow rule**.[19]

4.2.6 Specialization of the scalar flow rate d^p equation

For metallic materials a dependence on the stress invariants $\bar{p}$ and $\det \mathbf{M}_0^e$ has been experimentally found to be small. Accordingly, we neglect such a dependency here, and assume that the equivalent plastic shear strain rate is directly given by a flow function

$$d^p = \mathcal{F}(\bar{\tau}, \vartheta, \vec{S}) \geq 0 \quad \text{with} \quad \mathcal{F}(0, \vartheta, \vec{S}) = 0. \tag{4.57}$$

Alternatively, in many existing theories one assumes the existence of a **flow strength** function

$$\mathcal{S}(\vec{S}, d^p, \vartheta) > 0 \quad \text{whenever} \quad d^p > 0, \tag{4.58}$$

such that the equivalent shear stress satisfies the **flow condition**

$$\bar{\tau} - \mathcal{S}(\vec{S}, d^p, \vartheta) = 0 \qquad \text{whenever} \qquad \bar{\tau} > 0. \tag{4.59}$$

In this case (4.59) *serves as an implicit function to determine* d^p at a fixed state $\left\{\bar{\tau}, \vartheta, \vec{S}\right\}$.

4.2.7 Specialization of the evolution equations for the internal variables

We assume that at sufficiently high temperatures the internal variables may evolve not only when $d^p \neq 0$, but also when $d^p = 0$, and accordingly rewrite (4.56) as

$$\dot{S}_i = \underbrace{H_i(\vec{S}, d^p, \vartheta) d^p}_{\text{dynamic evolution}} - \underbrace{r_i(\vec{S}, \vartheta)}_{\text{static recovery}}\,, \tag{4.60}$$

In (4.60) H_i represent strain-hardening/softening function for the internal variables S_i during plastic flow, $d^p > 0$. The function $r_i \geq 0$ represent static thermal recovery functions for the internal variable S_i at a given temperature, whenever there is no macroscopic flow, $d^p = 0$.

[19] The classical Lévy–Mises–Reuss flow rule is a flow rule of this form for small deformations of rate-independent materials.

5 Final constitutive equations for a finite deformation rate-dependent theory for isotropic elastic-plastic materials with isotropic strain-hardening under isothermal conditions

1. **Kinematical decomposition of F**: the Kröner–Lee decomposition

$$\mathbf{F} = \mathbf{F}^e\mathbf{F}^p, \qquad \text{with} \qquad \det \mathbf{F}^p = 1, \tag{5.1}$$

in which $\mathbf{F}$ is the deformation gradient, while $\mathbf{F}^e$ and $\mathbf{F}^p$ are the elastic and plastic distortions.

2. **Free energy**:

With $\mathbf{F}^e = \mathbf{R}^e\mathbf{U}^e$ the polar decomposition of $\mathbf{F}^e$, $\{\lambda_i^e\}$ the positive eigenvalues and $\{\mathbf{r}_i^e\}$ the orthonormal eigenvectors of $\mathbf{U}^e$, and

$$\mathbf{E}^e \stackrel{\text{def}}{=} \sum_{i=1}^{3} (\ln \lambda_i^e)\, \mathbf{r}_i^e \otimes \mathbf{r}_i^e, \tag{5.2}$$

the logarithmic elastic strain tensor in the intermediate space,

$$\psi = G|\mathbf{E}_0^e|^2 + \tfrac{1}{2}K|\mathrm{tr}\,\mathbf{E}^e|^2, \tag{5.3}$$

where $G(\vartheta) > 0$ and $K(\vartheta) > 0$ are the elastic shear and bulk moduli, respectively.

3. **Equation for the stress**

The driving stress for plastic flow is the Mandel stress given by

$$\mathbf{M}^e = \mathbb{C}[\mathbf{E}^e] = 2G\mathbf{E}_0^e + K(\mathrm{tr}\,\mathbf{E}^e)\mathbf{1}, \tag{5.4}$$

where

$$\mathbb{C} \stackrel{\text{def}}{=} 2G\,(\mathbb{I} - \tfrac{1}{3}\mathbf{1}\otimes\mathbf{1}) + K\,\mathbf{1}\otimes\mathbf{1} \tag{5.5}$$

is the fourth-order isotropic elasticity tensor, with $\mathbb{I}$ the fourth-order identity tensor which maps symmetric tensors $\mathbf{A}$ into themselves $\mathbf{A} = \mathbb{I}[\mathbf{A}]$.

In terms of $\mathbf{M}^e$,

$$\bar{\tau} \stackrel{\text{def}}{=} \sqrt{1/2}\,|\mathbf{M}_0^e| \tag{5.6}$$

defines an equivalent shear stress.

The Cauchy stress in the deformed body is given by

$$\mathbf{T} = J^{e\,-1}\mathbf{R}^e\,\mathbf{M}^e\mathbf{R}^{e\top}, \qquad J^e = \det \mathbf{F}^e (\equiv \det \mathbf{F}). \tag{5.7}$$

4. **The flow rule**:

$$\left.\begin{aligned} \dot{\mathbf{F}}^p &= \mathbf{D}^p\mathbf{F}^p, \\ \mathbf{D}^p &= \sqrt{1/2}\,d^p\,\mathbf{N}^p, \\ \mathbf{N}^p &= \sqrt{1/2}\,(\mathbf{M}_0^e/\bar{\tau}), \end{aligned}\right\} \tag{5.8}$$

with the equivalent plastic shear strain rate either given directly by a flow function

$$d^p = \mathcal{F}(\bar{\tau}, \vartheta, \vec{S}) \geq 0 \quad \text{with} \quad \mathcal{F}(0, \vartheta, \vec{S}) = 0, \tag{5.9}$$

or alternatively, as a solution to an implicit flow equation

$$\bar{\tau} - \mathcal{S}(\vec{S}, d^p, \vartheta) = 0 \qquad \text{whenever} \qquad \bar{\tau} > 0, \tag{5.10}$$

where

$$\mathcal{S}(\vec{S}, d^p, \vartheta) > 0 \tag{5.11}$$

is a strain rate and temperature-dependent **flow strength** of the material.

5. **The evolution equations for the internal variables**:

$$\dot{S}_i = \underbrace{H_i(\vec{S}, d^p, \vartheta) d^p}_{\text{dynamic evolution}} - \underbrace{r_i(\vec{S}, \vartheta)}_{\text{static recovery}} \quad , \tag{5.12}$$

for the internal variables $\vec{S} = \{S_i \,|\, i = 1, n\}$ of the theory.

The evolution equations for $\mathbf{F}^p$ and $\vec{S}$ need be accompanied by initial conditions. Typical initial conditions presume that the body is initially (at time $t = 0$, say) in a **virgin state** in the sense that

$$\mathbf{F}(\mathbf{X}, 0) = \mathbf{F}^p(\mathbf{X}, 0) = \mathbf{1}, \qquad S_i(\mathbf{X}, 0) = S_{i,0} \text{ (= constant)}. \tag{5.13}$$