Summary of von Mises Yield Criterion

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Statement of yield criterion

Von Mises yield condition is [1]:

$$II_\sigma = k^2 \qquad (1)$$

where II_σ is the second invariant of deviatoric stress and k is a constant.

With this statement, we have enough information to relate k to a presumably known material property yield stress. In the next sections there is a review of relevant tensor algebra, formulae for second order tensor invariants, evaluation of yield criterion for a few examples and description of yield surface for a 2-D stress condition.

Invariants of a stress tensor[1]

Principal stresses of a symmetric tensor **T** can be found by:

$$\left|T - \lambda I\right| = 0 \qquad (2)$$

This equation is equivalent to the cubic equation:

$$\lambda^3 - I_T \lambda^2 - II_T \lambda - III_T = 0 \qquad (3)$$

where,

$$\begin{aligned} I_T &= T_{ii} = tr\mathbf{T} \\ II_T &= \frac{1}{2}\left(T_{ij}T_{ij} - I_T{}^2\right) \\ III_T &= \det \mathbf{T} \end{aligned} \qquad (4)$$

As the name suggests, the invariants of a tensor do not vary with the coordinate system selected.

Determination of yielding

As the criterion is on deviatoric stress, the mean normal stress (or equivalently pressure) component should be removed from the total stress tensor **T** :

$$\sigma_{ij} = T_{ij} - \sigma_h \delta_{ij} \qquad (5)$$

where δ_{ij} is Kronecker delta and mean hydrostatic stress $\sigma_h = \left(T_{11} + T_{22} + T_{33}\right)/3$. The second deviatoric stress invariant is:

$$II_\sigma = \sigma_{12}{}^2 + \sigma_{23}{}^2 + \sigma_{13}{}^2 - \sigma_{11}\sigma_{22} - \sigma_{22}\sigma_{33} - \sigma_{11}\sigma_{33} \qquad (6)$$

(5) and (6) can be combined to find this invariant in terms of total stress components

$$II_\sigma = \frac{1}{6}\left[\left(T_{11} - T_{22}\right)^2 + \left(T_{22} - T_{33}\right)^2 + \left(T_{33} - T_{11}\right)^2\right] + T_{12}{}^2 + T_{23}{}^2 + T_{13}{}^2 \qquad (7)$$

Now, we can calculate yielding for some examples.

<u>Uniaxial stress:</u> ($T_{11} = \sigma_y + C, T_{22} = T_{33} = C, T_{12} = T_{13} = T_{23} = 0$)
In this example, the material is loaded in one direction until yield stress, σ_y above the hydrostatic stress. Hydrostatic stress (or pressure) for an incompressible medium is indeterminate. Therefore, the constant C is introduced in the diagonal terms. Evaluated at this stress condition (7) reduces to:

$$II_\sigma = \frac{\sigma_y{}^2}{3} = k^2 \qquad (8)$$

Here, we see that $k = \sigma_y / \sqrt{3}$.

<u>Simple shear:</u> ($T_{12} = \tau_y, T_{11} = T_{22} = T_{33} = C, T_{13} = T_{23} = 0$)

$$II_\sigma = \tau_y{}^2 = k^2 \qquad (9)$$

From this example we see that k is yield stress in simple shear.

<u>Plane stress:</u> ($T_{33} = T_{13} = T_{23} = 0$)
For convenience we choose a coordinate system coinciding with the principal axes ($T_{12} = 0$) where $T_{11} = \sigma_1, T_{22} = \sigma_2$. The second invariant of the deviatoric stress is:

$$II_\sigma = \frac{1}{3}\left(\sigma_1{}^2 - \sigma_1\sigma_2 + \sigma_2{}^2\right) = k^2 \qquad (10)$$

Tresca yield criterion[1]

Tresca condition for yield is where maximum shear stress reaches yield stress. In terms of stress components this can be stated as:

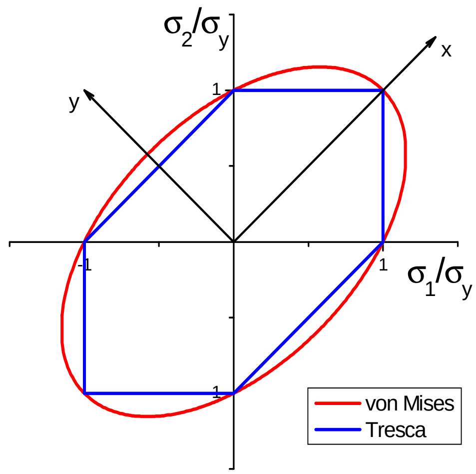
$$T_{\max} - T_{\min} = \sigma_y \qquad (11)$$

Yield surface

(10) can be simplified as:

$$\sigma_1^{\ 2} - \sigma_1\sigma_2 + \sigma_2^{\ 2} = \sigma_y^{\ 2} \qquad (12)$$

In graphical form the criterion is an ellipse in the principal stress plane as shown below.



We can make a coordinate transformation to convert the ellipse equation into a more familiar form:

$$\begin{aligned} \sigma_1 &= \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}} \\ \sigma_2 &= \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \end{aligned} \qquad (13)$$

The equation in x-y coordinates is:

$$\frac{x^2}{2\sigma_y^{\ 2}} + \frac{y^2}{\left(\frac{2\sigma_y^{\ 2}}{3}\right)} = 1 \qquad (14)$$

As it can be seen from the equation, the semimajor and the semiminor axes of the yield curve are $\sqrt{2}\sigma_y$ and $\sqrt{2/3}\sigma_y$ respectively. There are also two good resources on analytical algebra on ellipses available on the web[2,3].

References

[1] Malvern, L.E., "Introduction to the Mechanics of a Continuous Medium," Prentice Hall 1969

[2] Weisstien, Eric W., "Ellipse" From Mathworld – A Wolfram web resource. http://mathworld.wolfram.com/Ellipse

[3] Wikipedia, "Ellipse," http://en.wikipedia.org/wiki/Ellipse