# Scaling Relations in Elastic Turbulence

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We report the scaling relations between the exponents of the power-law decays of kinetic and elastic energies, pressure, as well as torque fluctuations in elastic turbulence (ET). The relations are derived by estimating that the divergent part of the elastic stress is much larger than its vortical part, and its contribution into the full elastic stress is dominant in the range of the power spectrum amplitudes observed experimentally in ET. The estimate is in line with polymer stretching by flow: the polymers are stretched mostly by the divergent part associated with a strain rate, whereas a rotational, or vortical, flow plays a minor role in the polymer stretching. The scaling relations agree well with the exponent values obtained experimentally and numerically in the ET regime of a viscoelastic fluid in different flow geometries.

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Elastic turbulence (ET) is a chaotic flow at low Reynolds number, $\mathrm{Re} \ll 1$, and large Weissenberg number, $\mathrm{Wi} = \lambda V/l \gg 1$, discovered two decades ago and currently well studied and characterized experimentally, theoretically, and numerically in various flow configurations with curvilinear streamlines. Here $\lambda$ is the longest polymer relaxation time, $V$ is the average flow velocity, and $l$ is the characteristic length scale. ET displays large velocity fluctuations in a wide range of spatial and temporal scales, a significant growth of flow resistance, and many orders of magnitude enhancement of mixing efficiency [1–3]. Elastic instabilities and ET at $\mathrm{Re} \ll 1$ are driven solely by elastic stresses generated by polymer stretching caused by the flow. The key observation in ET is the power-law decay of the kinetic energy spectra in a frequency domain $E(f) = \langle|\hat{u}(f)|^2\rangle \sim f^{-\alpha}$ with the exponent $\alpha > 3$ (between 3.3 and 3.6 depending on flow geometry) [1–4]. Since the validity of the Taylor hypothesis is justified with some limitations also in ET [5], one expects the same exponent value in both the $f$ and $k$ domains, that has been indeed confirmed via particle image velocimetry (PIV) in a swirling flow between rotating and stationary disks where $\alpha \simeq 3.5$ in the $k$ domain is measured [6].

Further on, additional key features of ET in a swirling flow are identified such as the power-law decays of the power spectra of torque $\Gamma$, $E_{\Gamma\Gamma} = \langle|\hat{\Gamma}(f)|^2\rangle$, and the pressure $p$, $E_{pp} = \langle|\hat{p}(f)|^2\rangle$. Both power spectra are characterized by a flat region at $f/f_{\mathrm{vor}} < 1$ and the steep power-law decay at $f/f_{\mathrm{vor}} > 1$ scaled as $E_{\Gamma\Gamma} \sim (f/f_{\mathrm{vor}})^{-\mu}$ [7,8] and $E_{pp} \sim (f/f_{\mathrm{vor}})^{-\beta}$ [7,8] at polymer concentration $\phi = 100$ ppm and various Wi and for various $\phi \leq 900$ ppm at the maximum Wi values with the exponents $\mu \simeq 4$ and $\beta \simeq 3$, respectively. Here $f_{\mathrm{vor}}$ is the main vortex frequency with a distinctive peak at $f/f_{\mathrm{vor}} = 1$ in the spectra [3,7,8]. One should emphasize a surprising repeatability and reliability of the value of $\beta \simeq 3$ obtained in ET in different flow geometries: a swirling flow between two disks, a wake between two widely separated obstacles hindering a channel flow, and a flow past a cylinder [4,7–10]. Moreover, in the swirling flow at $\phi = 300$ ppm and $\mathrm{Wi} = 100$ with more than $10^5$ data points taken in the separate experiment, one gets a very reliable result of $\beta = 3 \pm 0.1$ [see [8] Fig. 14(b)]. It is distinctly different from experimental values of $\alpha$ varied in the range 3.3–3.6 depending on flow geometry and in the range 3.2–3.8 in the numerical simulations.

One can show that in ET of a swirling flow, the torque is directly related to the averaged elastic stress, $\langle\sigma_{ij}\rangle$, on the driving disk. In general, $\Gamma(t) = \int_0^R \langle\tau_w(r,t)\rangle 2\pi r^2 dr$, where $\langle\tau_w(r,t)\rangle$ is the average stress on the driving disk. At $\mathrm{Re} \ll 1$ and Wi below an elastic instability onset, $\langle\tau_w(r,t)\rangle = \tau_w = \langle\sigma_{\mathrm{visc}}\rangle$, where $\langle\sigma_{\mathrm{visc}}\rangle$ is the averaged viscous shear stress on the disk that is independent of $(r,t)$ and related to the torque as $\Gamma = (2\pi R^3/3)\tau_w$. In the ET regime at $\mathrm{Re} \ll 1$ and $\mathrm{Wi} \gg 1$, $\langle\tau_w(r,t)\rangle = \langle\sigma_{ij}\rangle + \langle\sigma_{\mathrm{visc}}\rangle$. Moreover, in ET $\sigma_{\mathrm{visc}} \ll \sigma_{ij}$ and so $\langle\sigma_{\mathrm{visc}}\rangle \ll \langle\sigma_{el}\rangle$ [11,12]. Then one gets $\langle\tau_w(r,t)\rangle \approx \langle\sigma_{ij}(r,t)\rangle$. Thus in the approximation of the independence of $\langle\sigma_{ij}(t)\rangle$ on $r$, one obtains $\Gamma(t) \sim \langle\sigma_{ij}(t)\rangle$ that leads to $E_{\Gamma\Gamma} = \langle|\hat{\Gamma}(f)|^2\rangle \sim \langle|\hat{\sigma}(f)|^2\rangle \sim f^{-\mu}$.

To explain the power-law behavior and the scaling exponents of the main flow parameters in ET, theory [13] and numerical simulations [14,15] are carried out for an isotropic, homogeneous, and unbounded flow of a dilute polymer solution described by the Oldroyd-B model with linear polymer elasticity [16] and with a back reaction of the stretched polymers on the flow. Further on, the numerical simulations in a wall-bounded channel flow with

 

a linear periodic array of obstacles [17], and in two-dimensional Taylor-Couette flow geometry [18] of a dilute polymer solution characterized by the same Oldroyd-B model at low Re are employed to test the power-law decay exponent of the velocity power spectrum in ET. In all these simulations similar results of $\alpha > 3$ with variations between about 3.2 and 3.8 are found in accord with the experiment and theory. Another 2D direct numerical simulation (DNS) of forced fluid film flow of a polymer solution described by a finite extension nonlinear elasticity-Peterlin (FENE-P) polymer model [16] at $\mathrm{Re} \ll 1$ and $\mathrm{Wi} \gg 1$ reveals a chaotic flow characterized by the decay of the kinetic energy spectrum in a $k$ domain with $\alpha \approx 3.2$ [19], in a good agreement with the experiment [1,2], theory [13], and other numerical simulations [14,15,17,18] in ET. One of the key conclusions of DNS is the independence of the exponent value of the power-law decay of the kinetic energy spectrum on the polymer model.

The theory of ET is based on the Navier-Stokes equation (NSE) for a polymer solution with a coupling term, engendered by the back reaction of the elastic stress field $\sigma_{ij}$ on the velocity field $V_i$, and the constitutive equation for $\sigma_{ij}$ in the Oldroyd-B model approximation [16]. In ET at $\mathrm{Re} \ll 1$, an inertial term in NSE can be neglected. For sufficiently stretched polymers, one gets $\sigma_{ij} \gg \sigma^0_{ij}$, where $\sigma^0_{ij}$ is the elastic stress due to thermal noise, and two equations describing polymer hydrodynamics in the case of the Oldroyd-B model with the incompressibility condition are [16]

$$\nabla_i p = \eta \triangle V_i + F_i, \tag{1}$$

$$F_i = \nabla_j \sigma_{ij}, \tag{2}$$

$$\nabla_i V_i = 0, \tag{3}$$

$$d_t \sigma_{ij} = \sigma_{jk} \nabla_k V_i + \sigma_{ik} \nabla_k V_j - 2\sigma_{ij}/\lambda, \tag{4}$$

where $\eta$ is the dynamic viscosity. It was proved in [13] that in a statistically steady state realized at times much longer than the Lagrangian velocity gradient correlation time the polymer stress tensor $\sigma_{ij}$ is relaxed to the uniaxial form $\sigma_{ij} = B_i B_j$, if $\sigma_{ij} \gg \sigma^0_{ij}$, so $R_i \gg R_0$ is considered, where $R_i$ is the end-to-end vector for a given configuration of a polymer $i$, and $R_0$ is a radius of polymer random coil at equilibrium (gyration radius [16]). Then $\sigma_{ij} \sim \langle R_i R_j \rangle \sim R_i R_j$, as shown in [13]. The uniaxial form of $\sigma_{ij}$ allows us to rewrite the set of equations for polymer hydrodynamics in the form similar to magneto-hydrodynamic (MHD) equations [20] with $\sigma_{ij} = B_i B_j$ at zero magnetic resistivity [13], from which only the Stokes equation is presented below

$$\nabla_i p = \eta \triangle V_i + \nabla_j (B_i B_j), \tag{5}$$

$$\nabla_i V_i = 0, \tag{6}$$

$$\nabla_i B_i = 0, \tag{7}$$

where $B_i$ has the solenoidal nature, characterizes the degree of the coherent polymer extension, and, in a contrast to the magnetic field vector in MHD, is defined up to a sign, analogously to the director in nematic liquid crystals [13]. Another important difference with MHD appears in the relaxation term instead of the magnetic resistivity [20]. At the sufficiently large back reaction of the elastic stress determined by the term $(B_i \nabla_i) B_j$, the flow reaches a chaotic, statistically steady state defined as ET.

The major result of the theory based on Eqs. (1)–(4) is the prediction of the power-law decay of the spherically normalized kinetic energy spectrum $E(k) \sim V^2 \rho l (kl)^{-\alpha}$ with $\alpha > 3$ and the related to it the elastic energy spectrum $E_B(k) \sim B^2 l (kl)^{-\nu}$, where $\nu = (\alpha - 2) > 1$ that is close to the passive scalar Batchelor decay exponent $-1$ [21]. Here $k$ is the wave number, and $l$ is the length scale that defines the largest average velocity gradient. The exponents of both power spectrum decays are obtained by applying a passive scalar approach for a description of the small scale perturbations of the velocity $\vec{u}$ and elastic stress $\vec{b}$ fields passively advected by the large scale random velocity $V_i$ field, where $u_i,\ b_i \ll V_i, B_i$ and $\nabla_i u_j \ll \nabla_i V_J$, while $\nabla_i b_j \sim \nabla_i B_j$ [13]. From both of the power spectrum expressions, one finds that the main energy in ET is carried out by the stretched polymers resulted in the elastic energy $\sim B^2$. The latter follows from the relation $E(k) \sim \mathrm{Re} (kl)^{-2} E_B(k)$, where $\rho (V/B)^2 \sim \mathrm{Re} \ll 1$, since $B^2 \sim \eta/\lambda$ and $\rho V^2 \sim \rho (l/\lambda)^2$ that leads to $\rho (V/B)^2 \sim \rho l^2/\lambda\eta = \mathrm{El}^{-1} \sim \mathrm{Re}$ [13], where $\rho$ is the fluid density and the elasticity $\mathrm{El} = \mathrm{Wi}/\mathrm{Re} = \rho l^2/\lambda\nu \gg 1$, since in the experiment in the ET regime $\mathrm{Wi} \gg 1$ and $\mathrm{Re} \ll 1$. The theoretical value of $\alpha$ is found in good accord with the experimental results [1–4,6].

The same relation in ET between $\nu$ and $\alpha$ can also be obtained just by scaling arguments considering $\sigma_{ij} = B_i B_j \sim R_i R_j$ leading to $B_i \sim R_i$. A statistically steady state of a polymer extension is determined by the interplay between a stretching due the velocity difference between the polymer ends, estimated by the velocity gradient multiplied on $R_i$, and its relaxation. To simplify the issue, we consider a plane Couette flow with a uniform shear rate where a polymer is introduced. The polymer is stretched in the shear direction up to $R_\parallel$ and remains unchanged in the perpendicular one $R_\perp \approx R_0$ over the relaxation time $\lambda$ that leads to a relation $R_\parallel = R_0 \nabla \vec{V} \lambda$ (see [22]). Thus taking into account that $R_i \sim \nabla \vec{V}$, one finally obtains $E_B(f) \sim \langle \nabla \vec{V} \nabla \vec{V} \rangle_f \sim f^{-(\alpha-2)}$, since $E(f) \equiv \langle \vec{V}\, \vec{V} \rangle_f \sim f^{-\alpha}$, resulting in the equation $\nu = \alpha - 2$, equivalent to the relation

obtained by analytical calculations in [13] and discussed above.

Though the theory of ET predicts the first relation between the scaling exponents of the elastic and kinetic energy spectrum decays $\nu = \alpha - 2$ [13], it does not provide any clue on the relation between the scaling exponents of the kinetic energy spectrum and pressure power spectrum decays $\alpha$ and $\beta$, respectively.

Using the Stokes equation [see Eq. (5)], we derive in ET the relation between the Fourier transforms of the pressure $E_p(k) \sim k^{-\beta/2}$, velocity $E_u(k) \sim k^{-\alpha/2}$, and the elastic stress $E_B(k) \sim k^{-\nu}$, where $\nu = \alpha - 2$ is used,

$$iB^2(kl)(kl)^{-(\alpha-2)} = ip(kl)(kl)^{-\beta/2} + \eta(kl)Vl^{-1}(kl)^{-(\alpha-2)/2}. \tag{8}$$

The Eq. (8) presents the elastic stress power spectrum as a sum of two terms. By straightforward calculations we demonstrate that the first term on the right side of Eq. (8) is the power spectrum of the divergent part of the elastic stress $\sigma_{ij}^{\mathrm{div}} \equiv (B_i B_j)^{\mathrm{div}}$. Indeed, by taking a divergence operation on the both sides of Eq. (5), one gets $\triangle p = \partial^2 (B_i B_j)^{\mathrm{div}} / \partial x_i \partial x_j$ [7,8], which Fourier transform leads to the relation $E_B(kl)^{\mathrm{div}} \sim E_p(kl) \sim p(kl)^{-\beta/2}$ after canceling $(kl)^2$ in both terms. Then one finds that $E_B(kl)^{\mathrm{div}} \sim p(kl)^{-\beta/2}$ is equivalent to the first term on the right side of the Eq. (8) after canceling factor $kl$ in all three terms. Similarly, by taking a curl operation of the same Eq. (5) provides the exact relation between the vortical (divergent-free) part of the elastic stress $\sigma_{ij}^{\mathrm{curl}} \equiv (B_i B_j)^{\mathrm{curl}}$ and the vorticity $\vec{\Omega} \equiv \vec{\nabla} \times \vec{V}$: $\nabla \times \nabla (\vec{B}\vec{B})^{\mathrm{curl}} = -\eta \Delta \vec{\Omega}$ [8]. Then the Fourier transform results in the relation $E_B(kl)^{\mathrm{curl}} \sim \eta V l^{-1} (kl)^{-(\alpha-2)/2}$ that is equivalent to the second term on the right side of Eq. (8) after canceling the factor $kl$. Thus, the Eq. (8) can be rewritten as $E_B(kl) = E_B(kl)^{\mathrm{div}} + E_B(kl)^{\mathrm{curl}}$, and after substituting the expressions for $E_B(kl)^{\mathrm{div}}$ and $E_B(kl)^{\mathrm{curl}}$ and performing straightforward algebraic transformations one gets the following algebraic equation

$$B^2 x^2 = p x^{\beta/(\alpha-2)} - i(\eta V/l)x, \tag{9}$$

where $x = (kl)^{-(\alpha-2)/2}$. In ET at $kl < 1$ or $f\lambda < 1$ (and so $x > 1$) just before the power-law decay onset, the elastic energy spectrum is flat and the amplitude ratio of the Fourier transforms $E_B(kl)^{\mathrm{div}} / E_B(kl)^{\mathrm{curl}}$ is large due to both $x > 1$ and the large ratio of the coefficients of two terms on the right side of Eq. (9), namely $pl/V\eta \gg 1$. Regarding the second inequality, by neglecting the last term due it smallness one gets the equality $p = B^2$. Then by using $B^2 l / V\eta$, one can rewrite it as $(B^2/\rho V^2)(\rho l V/\eta) = \mathrm{El}(\rho l V/\eta) = \mathrm{El} \cdot \mathrm{Re} = \mathrm{Wi} \gg 1$. Then the ratio $E_B(kl)^{\mathrm{div}} / E_B(kl)^{\mathrm{curl}} \gg 1$ at $kl < 1$ and even at $kl > 1$ up to at least $kl \approx 10$ in spite of a decay of $E_B(kl)^{\mathrm{div}}$ steeper than of $E_B(kl)^{\mathrm{curl}}$ at $kl > 1$ (or $f\lambda > 1$). Thus, the last term on the right side of Eq. (9) can be neglected in the whole range of the power-law decay of the energy spectrum obtained in the experiment. As the result, one gets a new scaling relation:

$$\beta = 2(\alpha - 2). \tag{10}$$

Thus, the conclusion is that $\sigma_{ij}^{\mathrm{div}}$ plays the dominant role in the elastic stress field, whereas $\sigma_{ij}^{\mathrm{curl}}$ has the minor contribution into polymer stretching, and so the vortical part of the elastic stress in ET can be neglected in the range of the velocity and pressure power spectra found experimentally [1–4,6–10]. Moreover, this conclusion is in line with the physics of the polymer stretching in ET: polymers are considerably stretched by the flow strain rate reflected by the divergent part of the elastic stress, whereas in rotational (vortical) flow polymers remain almost unstretched. This conclusion is recently confirmed and clarified both theoretically and numerically by considering a novel effect of preferential sampling of elastic chains in turbulent flows. It is demonstrated numerically that, first, the stretched chains are located in the flow regions with low vorticity, and, second, the elastic chains are trapped in the vortical regions, where they are found in a coil state with the negligible contribution into the elastic energy [23].

Using the new scaling relation [Eq. (10)] and the experimental value of $\alpha \approx 3.5$, one gets for the power-law decay exponent of the pressure power spectrum $\beta \approx 3$ in good agreement with the experiment [4,7–10], and for the algebraic decay exponent of the elastic stress power spectrum $\nu \approx 1.5$, which is still not verified in ET either experimentally or numerically. On the other hand, from the above suggested relation between the torque and averaged over the upper driving disk elastic stress power spectra $E_{\Gamma\Gamma} = \langle |\hat{\Gamma}(f)|^2 \rangle \sim \langle |\hat{\sigma}(f)|^2 \rangle \sim f^{-\mu}$ one gets the relation $\mu = 2\nu$. In this case, taking the obtained above value $\nu \approx 1.5$, one finds $\mu \approx 3$ instead of the experimental value $\mu \simeq 4$ [7,8], where the discrepancy could be attributed to the assumptions, first that $\langle \sigma_{ij}(r,t) \rangle$ is independent of $r$ to get the relation $\Gamma(t) \sim \langle \sigma_{ij}(t) \rangle$ and second that the scaling exponent for $\sigma_{ij}$ and $\langle \sigma_{ij} \rangle$ is the same used in the derivation of $\mu = 2\nu$.

Another well-known example of a chaotic, spatially smooth and random in time flow is a high-Re inertial turbulence in a deep-dissipation scale range, in Newtonian fluid turbulence. In this case at scales $k^{-1} < \zeta$, the kinetic energy spectrum decays exponentially, and the velocity field is characterized by the single spatial scale $\zeta$ [24]. There is significant evidence coming from 3D DNS of homogeneous and isotropic viscoelastic turbulence in a turbulent drag reduction (TDR) regime that the fluid elasticity qualitatively modifies the velocity power spectrum at scales $k^{-1} < \zeta$ due to a strong increase of a kinetic energy content caused by polymer stretching [22,25–30].

It has been demonstrated by 3D DNS of a simplified viscoelastic fluid model that in the inertial range, at large scales, a turbulent energy cascade leading to the kinetic energy power spectrum remains roughly unchanged by the presence of polymers, whereas at small scales, below $\zeta$, the velocity power spectrum decay is modified from exponential for Newtonian fluid to algebraic for viscoelastic fluid, though the flow remains chaotic in time and spatially smooth [22,25–30].

The first quantitative results on the power-law decay $k^{-\alpha}$ of the kinetic energy power spectrum with a detailed analysis of the dependence of $\alpha$ on Wi and Re are obtained by numerical calculations using a hybrid Eulerian-Lagrangian approach to 3D homogeneous, isotropic, decaying turbulence in a polymer solution characterized by the FENE-P constitutive equation and at rather low initial Re (or at the turbulent Reynolds number $\mathrm{Re}_\lambda(t* = 0) = 52$ based on the Taylor microscale) to attain larger $\zeta$ value [see Figs. 5, 6 [27], Fig. 30(a) [28], and Figs. 5, 6 in [29]]. At the reduced time $t* = 20$ with $\mathrm{Re}_\lambda(t* = 20) = 3.5$, 3.2, 3.1, 3.0, and at four values of $\mathrm{Wi} = 25$, 50, 100, 200, it is found that $\alpha$ decreases from 4.6 down to 4.1 in the deep-dissipation range of scales, as Wi increases from 25 up to 200 [27,29]. However, the power spectrum amplitudes increase with Wi and significantly exceed that of Newtonian fluid [28]. Moreover, the same calculations reveal the pressure and elastic energy power spectrum decays with the exponents $\beta = 3 \pm 0.2$ and $\nu = 2.2 \pm 0.1$, respectively [29]. The authors of Ref. [29] have also undertaken an attempt to derive a scaling relation between $\alpha$ and $\beta$, though the suggested relation, which is different from that discussed above in this Letter, is not consistent with their numerical results.

Later on, a role of polymers in altering small scale dynamics of homogeneous, isotropic turbulence at $\mathrm{Re} \gg 1$ is investigated by 3D DNS, and a short review of former relevant numerical results and a possible connection to ET are provided in Ref. [22]. The early results [25,26,30,31] on the small scale dynamics in TDR at $\mathrm{Re} \gg 1$ in the presence of polymers are presented in a qualitative way only. So the early studies are conducted via 3D DNS in homogeneous, isotropic, stationary turbulence at moderate $\mathrm{Re}_\lambda \simeq 87$ using the linear polymer Oldroyd-B model. They reveal that the kinetic energy spectrum is partially suppressed by energy transfer to polymers at $k^{-1} < \zeta$, contrary to a significant increase found in other studies [26,30,32], and the kinetic energy spectrum demonstrates an algebraic decay with $\alpha \approx 3.5$ instead of an exponential decay for Newtonian fluid [25]. The value of $\alpha$ is close to that found in other studies [25,26,30]. Another 3D DNS calculations carried out at low elasticity and moderate Re in homogeneous, isotropic, decaying turbulence using the FENE-P polymer model report a significant decrease of the dissipation energy rate and an amplification of the kinetic energy and enstrophy spectrum amplitudes in the deep-dissipation range together with a slight simultaneous decrease of the kinetic energy spectrum in the inertial range [30]. Moreover, the kinetic energy spectrum shows power-law decay with $\alpha \approx 3.5$ at $k^{-1} < \zeta$, close to that mentioned above [25]. Further progress in the DNS studies, where the scale-by-scale equation of the interaction between polymers and fluid flow is exploited, is presented in [31] and then later used in [33], where the scale-by-scale kinetic energy flux is analyzed. The latter reveals that the polymers remove energy from large scales and then transfer to the small scales where it dissipates. The energy balance between scales is further studied in [22], where the power-law decay exponent $\nu \simeq 2$ of the elastic energy spectrum is clearly verified. The latter agrees well with the finding in [29,31] and is independent of the polymer model used. Recently the first experiment in a large facility of grid turbulence in dilute polymer solutions at $\mathrm{Re} = 2 \times 10^4$ reveals the exponent $\alpha \simeq 3$ of the power-law decay of the velocity spectrum at scales close but still larger than $\zeta$ [34]. It is the first verification of various numerical results in the TDR regime at the small scales.

To summarize the results of the DNS investigations of homogeneous, isotropic turbulence at scales $k^{-1} < \zeta$, one concludes that the values of the scaling exponents $\alpha$, $\beta$, and $\nu$ are rather close to those found in ET and independent of the polymer model. However, a larger scatter in values for $\alpha$ is found: between $\alpha \approx 3.5$ in DNS with various constitutive equations [25,26,30] in a good agreement with ET versus a hybrid Eulerian-Lagrangian DNS $\alpha$ between 4.1 and 4.6 depending on Wi [27–29]. Such scatter in values of $\alpha$ leads to the discrepancy between values of $\beta$ and $\nu$ obtained in DNS compared with those found from the scaling relations.

To conclude, the relations between the scaling exponents $\alpha$, $\nu$, $\beta$, $\mu$ of $\vec{V}$, $\vec{B}$, $p$, and $\Gamma$, are suggested and summarized in Table I. The relation between $\alpha$ and $\beta$ is well satisfied by the exponent values taken from either experiments or DNS in ET in various flow geometries, whereas the agreement between $\alpha$, $\nu$, and $\beta$ from DNS and the values obtained from the two scaling relations applied to high-Re viscoelastic turbulence at scales $k^{-1} < \zeta$ is less satisfactory. This discrepancy between ET and high-Re viscoelastic turbulence may be caused by a different nature of the elastic stress field, namely the energy flux from the inertial to dissipation scales in latter versus its absence in ET. It can lead, for example, to a larger contribution of the vortical part of the elastic stress field $E_B(kl)^{\mathrm{curl}}$ in Eq. (9) than in ET, where $E_B(kl)^{\mathrm{curl}}$ is neglected in the derivation of the scaling relation in Eq. (10).

TABLE I. Table of the scaling exponent relations.

| $\beta$ | $\nu$ | $\mu$ |
|---|---|---|
| $2(\alpha-2)$ | $\alpha-2$ | $2(\alpha-2)$ |

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[1] A. Groisman and V. Steinberg, Nature (London) **405**, 53 (2000).
[2] A. Groisman and V. Steinberg, Nature (London) **410**, 905 (2001).
[3] A. Groisman and V. Steinberg, New J. Phys. **6**, 29 (2004).
[4] A. Varshney and V. Steinberg, Phys. Rev. Fluids **3**, 103303 (2018).
[5] T. Burghelea, E. Segre, and V. Steinberg, Phys. Fluids **17**, 103101 (2005).
[6] T. Burghelea, E. Segre, and V. Steinberg, Phys. Fluids **19**, 053104 (2007).
[7] Y. Jun and V. Steinberg, Phys. Rev. Lett. **102**, 124503 (2009).
[8] Y. Jun and V. Steinberg, Phys. Rev. Fluids **2**, 103301 (2017).
[9] D. Li, A. Varshney, and V. Steinberg, Phys. Rev. Fluids (to be published).
[10] A. Varshney and V. Steinberg, Phys. Rev. Fluids **3**, 103302 (2018).
[11] A. Groisman and V. Steinberg, Phys. Rev. Lett. **86**, 934 (2001).
[12] Y. Liu and V. Steinberg, Europhys. Lett. **90**, 44002 (2010).
[13] A. Fouxon and V. Lebedev, Phys. Fluids **15**, 2060 (2003).
[14] S. Berti, A. Bistagnino, G. Boffetta, A. Celani, and S. Musacchio, Phys. Rev. E **77**, 055306(R) (2008).
[15] S. Berti and G. Boffetta, Phys. Rev. E **82**, 036314 (2010).
[16] R. B. Bird, R. C. Armstrong, and O. Hassager, *Dynamics of Polymeric Liquids: Fluid Mechanics*, 2nd ed. (John Wiley & Sons, New York, 1987), Vols. 1 and 2.
[17] M. Grilli, A. Vázquez-Quesada, and M. Ellero, Phys. Rev. Lett. **110**, 174501 (2013).
[18] R. van Buel, C. Schaaf, and H. Stark, Europhys. Lett. **124**, 14001 (2018).
[19] A. Gupta and R. Pandit, Phys. Rev. E **95**, 033119 (2017).
[20] L. D. Landau and E. M. Lifschitz, *Electrodynamics of Continuous Media* (Pergamon, Oxford, 1989).
[21] G. K. Batchelor, J. Fluid Mech. **5**, 113 (1959).
[22] M. Q. Nguyen, A. Delache, S. Simoens, W. J. T. Bos, and M. El Hajem, Phys. Rev. Fluids **1**, 083301 (2016).
[23] J. R. Picardo, D. Vincenzi, N. Pal, and S. S. Ray, Phys. Rev. Lett. **121**, 244501 (2018).
[24] L. D. Landau and E. M. Lifschitz, *Fluid Mechanics* (Pergamon, Oxford, 1987).
[25] S. Berti, A. Bistagnino, G. Boffetta, A. Celani, and S. Musacchio, Europhys. Lett. **76**, 63 (2006).
[26] P. Perlekar, D. Mitra, and R. Pandit, Phys. Rev. Lett. **97**, 264501 (2006).
[27] T. Watanabe and T. Gotoh, J. Phys. Conf. Ser. **454**, 012007 (2013).
[28] T. Watanabe and T. Gotoh, J. Fluid Mech. **717**, 535 (2013).
[29] T. Watanabe and T. Gotoh, Phys. Fluids **26**, 035110 (2014).
[30] P. Perlekar, D. Mitra, and R. Pandit, Phys. Rev. E **82**, 066313 (2010).
[31] E. De Angelis, C. M. Casciola, and R. Piva, Phys. D **241**, 297 (2012).
[32] L. Thais, G. Mompean, and T. B. Gatski, J. Non-Newtonian Fluid Mech. **200**, 165 (2013).
[33] P. C. Valente, C. B. de Silva, and F. T. Pinho, J. Fluid Mech. **760**, 39 (2014).
[34] R. Vonlanthen and P. Monkewitz, J. Fluid Mech. **730**, 76 (2013).